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WENTWORTH-SMITH MATHEMATICAL SERIES

SCHOOL ALGEBRA

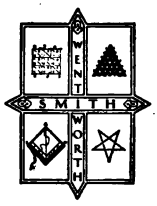
BOOK II

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH



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BII

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FROM THE [illegible]
CHARLES HERBERT [illegible]
Jul 31, 1928

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PREFACE

In the preparation of this work the authors have consulted the courses of study in general use in the various states and leading cities of this country, and have considered with great care the syllabi and suggested curricula prepared by the various important associations of teachers of mathematics. They have also studied the papers recently set by the principal examination boards and have taken the judgment of a large number of prominent teachers as to the best selection and arrangement of topics for the first and second years' work in high-school algebra. As a result of this careful investigation, extending over a considerable period of time, they have arranged a work in two books — the first covering the topics generally agreed upon as suited to the pupil's ninth school year; and the second offering a review of the first book and carrying the subject through the Binomial Theorem, Logarithms, and the theory of Complex Numbers.

Book II is intended to be used in the second year of algebra. It is so arranged as to furnish a whole year's work if desired, but teachers will find it a simple matter to select only enough material for a half year. It is a matter of general experience that pupils forget their first year's work before taking the subject up again, and for this reason a thorough review of this part of algebra has been prepared. This review may be omitted entirely, in which case a half year may be spent on the rest of the work; or only such parts of the review may be selected as are most needed, part of the subsequent problems being omitted to allow for this selection. In any case enough drill work is provided so that a teacher may select and change the material from year to year.

The applied problems include as many as possible of the kind met in real life, these necessarily being supplemented by others that have their chief value in the interest they arouse and the drill they give. The topics are those that are found in all standard courses of study extending through the Binomial Theorem. Logarithms and the graphic representation of the Complex Number are also included, to be taken or not, as circumstances permit.

For obvious reasons a few of the definitions and illustrations of Book I have been repeated. In general, however, all the review work is on a somewhat higher plane of difficulty than is found in the corresponding part of Book I. The chapter on Ratio and Proportion is given in both books, as is part of the work on graphs and part of the theory of quadratics, some teachers preferring to include these topics in the first year's work, and some in the second. The authors feel that this flexibility of arrangement will be approved by all teachers.

The attempt has been made to set forth, in simple language, the modern idea of function, without carrying the work to an unwarranted extreme. The authors have presented the graph in the same spirit, introducing it gradually and with the definite purpose of leading the pupil to recognize, through visual aids, the nature of negative numbers and the number and nature of the roots of equations.

Particular attention is called to the Cumulative Review in the Appendix. This furnishes the opportunity for a careful review of all the preceding work at the end of each chapter, and the value of the plan will be apparent to every teacher.

The authors wish to express their thanks to the many teachers who have assisted them either by their valuable suggestions or by their care in the reading of the proof. Any suggestions for further improvement of the work will be gratefully received.

GEORGE WENTWORTH
DAVID EUGENE SMITH

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SCHOOL ALGEBRA

BOOK II

CHAPTER I

REVIEW OF THE FUNDAMENTAL OPERATIONS

1. Uses of Algebra. From the study of Book I it has been seen that algebra is of importance to every one who has to make any use of formulas, that it is helpful in the solution of practical problems, that it is interesting as a science by itself, and that it forms a necessary preliminary to the study of higher mathematics.

2. Need for Review. The student taking up Book II is familiar with the ordinary symbols and terms used in algebra, but has probably forgotten many important details. It is therefore necessary to review briefly the work of Book I before proceeding to more advanced topics.

It is not necessary that exact definitions should be recalled for such familiar words as "monomial" and "exponent," but the student should be able to illustrate all such terms and to explain their meaning.

3. Passing from Words to Symbols. One of the great values of algebra is its power of expressing words by symbols.

For example, instead of writing "the square of some number added to four times the number, and the sum diminished by 7," we may write $x^2 + 4x - 7$, or $n^2 + 4n - 7$.

To express algebraically five times any number, we may write $5n$, $5N$, or $5x$, etc. Capital letters are often used, particularly in formulas.

Exercise 1. From Words to Symbols

Examples 1 to 3, oral — Examples 4 to 32, written

1. Express algebraically three less than twice any number.

2. Express algebraically the sum of any two numbers; twice the sum of any two numbers.

3. Express algebraically the difference between any two numbers; three times the difference; half of the difference.

Write in figures, using the appropriate signs:

4. 6 times the sum of 4 and 5.

5. 5 times the difference between 17 and 8.

6. $\frac{2}{3}$ of the sum of 9, 15, 18, 27, 33, and 36.

Express algebraically:

7. n times the sum of a and 3; of a and $3b$.

8. n times the difference between 19 and 7; between a and b.

9. 7 less than twice a; than half of a; than a times b.

10. y more than the quotient of a divided by 5.

Express the cost of:

11. 3 things at 5¢ each; n things at 5¢ each.

12. n things at c cents each; at $3c$ cents each.

13. n things at 60¢ a dozen; at c cents a dozen.

14. n things when a of them cost c cents.

15. x things at 4 for 25¢; at n for c cents; at x for 3¢.

State the distance traveled in:

16. 3 hr. at 3 mi. per hour; h hours at m miles per hour.

17. T hours at 45 mi. per hour; at m miles per hour.

18. t hours at 1 mi. in 2 min.; at M miles in H hours.

19. m minutes at 4 mi. per hour; at a miles in b hours; at a miles in $b + \frac{1}{2}$ hours.

Express algebraically the following statements:

20. The square of the sum of two numbers; the sum of the squares of the numbers.

21. The square of the difference of two numbers; the difference of the squares of the numbers.

22. The sum of two numbers is equal to their difference, plus twice the smaller number.

23. The sum of two numbers plus their difference is equal to twice the larger number.

24. The sum of the squares of two numbers is equal to twice the product of the numbers, plus some other number.

25. The weight in pounds of a pine beam with a square end is 40 times the length multiplied by the square of the thickness, the foot being the unit.

26. The weight in pounds of a cube of granite is 168 times the third power of its edge, the foot being the unit.

27. The square of the sum of two numbers equals the sum of their squares, together with twice their product.

28. The square of the difference of two numbers equals the sum of their squares, less twice their product.

29. The product of the sum of two numbers and the difference of the numbers equals the difference of the squares of the numbers.

30. The sum of two numbers multiplied by the larger number equals the square of the larger number, together with the product of the two numbers.

31. The sum of three numbers multiplied by a fourth number equals the sum of the products of each of the three numbers by the fourth number.

32. The sum of three numbers multiplied by one of the numbers equals the square of the one number added to the products of this number by each of the others.

Exercise 2. Review of Common Terms

Examples 1 to 27, oral — Examples 28 to 32, written

1. In $2x^3$, what is the numerical coefficient? the exponent?
2. In $-\frac{3}{5}x$, what is the coefficient of x? the exponent?
3. State the distinction between coefficient and exponent?
4. What is an algebraic formula? Illustrate.
5. What are the terms of a polynomial? Illustrate.

Explain the meaning of the following terms and give an illustration of each:

6. Binomial.
7. Trinomial.
8. Polynomial.
9. Evaluation.
10. Equation.
11. Axiom.
12. Negative number.
13. Absolute value.
14. Factor.
15. Similar terms.
16. Degree of a term.
17. Homogeneous.
18. Identity.
19. Root of an equation.
20. Quadratic trinomial.
21. Common factor.
22. Fraction.
23. Mixed expression.
24. Reciprocal.
25. Proportion.

26. Absolute term of a polynomial.

27. Absolute term of an equation.

28. Write five illustrations of the use of a negative number.

29. Write a polynomial of the fifth degree arranged according to the descending powers of some letter.

30. Write a polynomial of the sixth degree arranged according to the ascending powers of some letter.

31. Write a homogeneous polynomial of the third degree.

32. Give three illustrations of the use of symbols of aggregation.

4. Common Formulas of Mensuration. There are certain formulas that are commonly learned in arithmetic, and which the student may also have met in algebra or geometry. It is desirable to review these and to state them for future reference.

1. *Rectangle:* $a = bh$.

In general, initial letters are used to represent the numerical values of magnitudes. Thus a represents the number of square units of area, b the number of linear units in the base, and h the number of linear units in the altitude (height).

This may be briefly stated in words, "the area equals the product of the base and altitude (height)."

2. *Parallelogram:* $a = bh$.

3. *Triangle:* $a = \frac{1}{2}\, bh$.

4. *Trapezoid:* $a = \frac{1}{2}(b + b')\, h$.

That is, the area of a trapezoid equals half the product of the sum of the bases by the altitude.

5. *Circumference of a circle:* $c = \pi d$, or $c = 2\,\pi r$.

Here d stands for diameter and r for radius. The symbol π (pī) stands for 3.14159 +, or nearly 3.1416. A convenient value for practical use is $3\frac{1}{7}$, or $\frac{22}{7}$.

6. *Area of a circle:* $a = \frac{1}{2}\, cr$, $a = \pi r^2$, or $a = \frac{1}{4}\, \pi d^2$.

7. *Volume of a rectangular solid:* $v = lwh$.

Here l stands for the length, w for the width, and h for the altitude.

8. *Volume of a prism:* $v = bh$.

Here b stands for the number of square units in the base, and h for the number of linear units in the altitude.

9. *Volume of a pyramid:* $v = \frac{1}{3}\, bh$.

10. *Volume of a cylinder:* $v = bh$.

If the base is a circle, $v = \pi r^2 h$, or $\frac{1}{4}\, \pi d^2 h$.

11. *Volume of a cone:* $v = \frac{1}{3}\, bh$.

If the base is a circle, $v = \frac{1}{3}\, \pi r^2 h$, or $\frac{1}{12}\, \pi d^2 h$.

12. *Surface of a sphere:* $s = 4\,\pi r^2$, or $s = \pi d^2$.

13. *Volume of a sphere:* $v = \frac{4}{3}\, \pi r^3$, or $v = \frac{1}{6}\, \pi d^3$.

Exercise 3. Evaluation of Expressions

Examples 1 to 9, oral — Examples 10 to 33, written

1. Evaluate $a + b$ for $a = 7, b = 5$; for $a = 9, b = 11$.

2. Evaluate $2a + b$ for $a = 1, b = 5$; for $a = 5, b = 5$.

3. Evaluate $2a + 3b$ for $a = 1, b = 1$; for $a = 2, b = 1$.

4. Evaluate $a - b$ for $a = 7, b = 5$; for $a = 9, b = 7$.

5. Evaluate $4a - b$ for $a = 1, b = 1$; for $a = 2, b = 1$.

6. Evaluate ab for $a = 2, b = 3$; for $a = \frac{1}{2}, b = 12$.

7. Evaluate $a \div b$ for $a = 6, b = 2$; for $a = 9, b = 3$.

8. Evaluate $2n$ for $n = 1$; for $n = 7$; for $n = 12$; for $n = 47$.

9. Evaluate $2n + 1$ for $n = 1$; for $n = 7$; for $n = 59$.

Copy and write the values:

10. $3n$; $n = 7, 5, 9, 39$.

11. $3n + 1$; $n = 7, 5, 11, 77$.

12. $3n - 1$; $n = 5, 7, 16, 28$.

13. $5n$; $n = 17, 39, 179$.

14. $5n + 1$; $n = 26, 37, 79$.

15. $5n - 1$; $n = 43, 72, 67$.

16. $6n$; $n = 32, 48, 79, 87$.

17. $6n + 1$; $n = 75, 97, 128$.

18. $7n + 1$; $n = 36, 42, 88$.

19. $7n - 1$; $n = 44, 78, 96$.

Taking $\pi = 3.1416$, evaluate the following:

20. $2\pi r$; $r = 15$.

21. πr^2; $r = 12$.

22. $\pi r^2 h$; $r = 1, h = 3$.

23. $4\pi r^2$; $r = 12$.

24. $\frac{4}{3}\pi r^3$; $r = 9$.

25. $2\pi r$; $r = 37$.

26. πr^2; $r = 75$.

27. $\pi r^2 h$; $r = 2, h = 7$.

28. $4\pi r^2$; $r = 7\frac{1}{2}$.

29. $\frac{4}{3}\pi r^3$; $r = 12$.

30. If $x = 21$, find the values of $x + 7$; $x - 7$; $7x$; $x + 7$.

31. If $x = 51$, find the values of $96 + x$; $96 - x$; $96x$.

32. If $x = 75$, find the values of $x + 25$; $x - 25$; $25x$.

33. A rectangular subway is to be made w feet wide and h feet high. The earth weighs p pounds per cubic foot. How many tons of earth will be removed in l feet of length? Evaluate the result for $w = 35, h = 16, l = 1000, p = 70$.

Exercise 4. Formulas and Problems

Examples 1 to 4, oral — Examples 5 to 33, written

1. State in your own language the rule for finding the area of a rectangle. State this as a formula.

2. State the rule and also a formula for finding the area of a triangle.

3. State the rule and also a formula for finding the area of a parallelogram.

4. State the rule and also a formula for finding the circumference of a circle.

Write the formulas for the rules in Exs. 5–10, using initial letters, and evaluate for the given values of the letters:

5. The volume of a prism equals the product of the base and height. (Let $b = 27$, $h = 3\frac{1}{2}$.)

6. The volume of a cube equals the third power of the edge. (Let $e = 21.3$.)

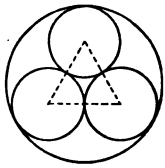
7. The volume of a pyramid equals one third the product of the base and height. (Let $b = 48$, $h = 35$.)

8. The volume of a cylinder equals π times the square of the radius, multiplied by the height. (Let $r = 1$, $h = 2$.)

9. The volume of a cone equals one third of π times the square of the radius of the base, multiplied by the height. (Let $r = 10$, $h = 30$, $\pi = 3.1416$.)

10. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius. (Let $r = 2$. Also solve for $r = 10$.)

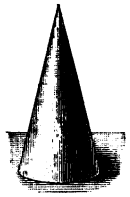
11. Evaluate the formula of Ex. 10 for $r = 21$, $\pi = 3\frac{1}{7}$; for $r = 35$, $\pi = 3\frac{1}{7}$.



12. If the radius of the large circle in this figure is R and the radius of each of the small circles is r, write a formula for the area of that part of the large circle not covered by the small circles. Evaluate for $R = 14$, $r = 7$; for $R = 10$, $r = 5$.

13. The volume of a cone equals one third the product of 3.1416 times the square of the radius of the base, multiplied by the height. Write the formula and evaluate for $r = 3$, $h = 5$.

14. The formula for the lateral surface of a cone in terms of the circumference (c) of the base and the distance (k) from the vertex to the circumference of the base is $l = \frac{1}{2}kc$. Write this as an ordinary sentence.



15. The formula for the lateral surface of a cone in terms of the radius of the base and the distance from the vertex to the circumference of the base is $l = \pi kr$. Write this as an ordinary sentence.

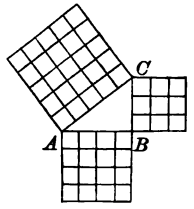
16. The formula for the surface of a sphere is $s = 4\pi r^2$. Write this as an ordinary sentence.

17. How many square inches in the surface of a sphere of which the radius is 7 in.?

18. The formula for the volume of a sphere is $v = \frac{4}{3}\pi r^3$. Write this as an ordinary sentence.

19. How many cubic inches in the volume of a sphere of which the radius is 7 in.?

20. The square on the hypotenuse of a right triangle is, as seen in the following figure, equal to the sum of the squares on the other two sides. Letting $AB = x$, $BC = y$, and $AC = z$, we have $z^2 = x^2 + y^2$, or $z = \sqrt{x^2 + y^2}$. Suppose $x = 4$ and $y = 3$, find the value of z.



21. Given $z = \sqrt{x^2 + y^2}$, find the value of z when $x = 24$ and $y = 32$.

22. In the same formula find the value of z when $x = 36$ and $y = 27$.

23. In the same formula find the value of z when $x = 33$ and $y = 44$.

24. In the same formula find the value of z when $x = 48$ and $y = 36$; when $x = 5.1$ and $y = 6.8$; when $x = 0.51$ and $y = 0.68$.

25. If an automobile has a constant rate of 12 mi. an hour, how far will it go in $3\frac{1}{2}$ hr.? Write a formula for d, the distance that it will go in t hours, at v miles per hour.

26. What is the volume of a box 12 in. long, $8\frac{1}{2}$ in. wide, and 6 in. deep? Write a formula for v, the volume of a box that is l inches long, w inches wide, and d inches deep.

27. The formula for the radius of a circle in terms of the circumference is $r = \frac{c}{2\pi}$. Write this as an ordinary sentence.

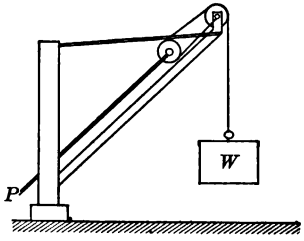
28. If it is c inches around an iron pipe, the diameter is $\frac{c}{\pi}$ inches; that is, $d = \frac{c}{\pi}$. Write this as an ordinary sentence.

29. The circumference of a water pipe is 22 in. Using the formula in Ex. 28, and taking $3\frac{1}{7}$ as the value of π, find the diameter.

30. The circumference of a wheel is 13.2 ft. Find the diameter. Find the radius.

31. A man who runs a stationary engine for hoisting iron reads in a book about engines that if s is the area of the outside shell of his boiler and h is the heating surface, then $h = \frac{2}{3}s$. What does this mean? What is the value of h if $s = 120\frac{5}{8}$ sq. ft.?

32. A foreman of a shop has a hoist like this for lifting heavy weights. He reads in a trade journal that in order to lift a weight W, the power P must be such that $P = 2W$. What power must his engine apply to the cable so as to lift a weight of 18,000 lb.?



33. The volume of a cylinder being represented by $v = \pi r^2 h$, what will the water in a cylindrical water tank weigh when $r = 7$, $h = 10$, $\pi = 3\frac{1}{7}$, the dimensions being in feet and the weight of 1 cu. ft. of water being $62\frac{1}{2}$ lb.?

5. Review of Addition. Addition in algebra is similar to addition in arithmetic except for the negative number.

The result obtained by adding two or more numbers considered with respect to their signs as well as their values is called their *algebraic sum.*

Thus the algebraic sum of 2, -3, 7, and -4 is 2.

6. Addition of Monomials. *To add similar monomials, find the algebraic sum of the coefficients of the common factor and prefix this sum to the common factor.*

For this purpose we may consider an expression like $27(a-b)$ as a monomial. In case a letter has no coefficient, 1 is understood.

To add dissimilar monomials, write the terms one after the other, each with its proper sign.

Thus the sum of a, $2b$, and $-c$ is $a+2b-c$.

7. Addition of Polynomials. It is evident that

To add polynomials we write similar terms in the same column and add these terms, writing their sums as a polynomial.

8. Check. An operation that tends to prove the correctness of another operation is called a *check* upon that operation.

9. Checks in Algebra. One of the best checks on the operations in algebra is the substitution of any values we please for the letters, as in the following example, where we let $x=1$, $y=1$, and $z=1$.

Operation	Check
$2x+3y-4z+6$	$2+3-4+6=7$
$3x-7y+8z-3$	$3-7+8-3=1$
$5x-4y+4z+3$	$5-4+4+3=8$

10. Commutative Law. Since $2+3=3+2$, so $a+b=b+a$, and $a+b+c=c+a+b$. That is, we may change the order of the quantities added without changing the sum. This is known as the *commutative law of addition.*

Exercise 5. Addition

1. Draw three lines representing the lengths a, b, and c. Draw lines representing $a+b+c$, $a+c+b$, and $c+b+a$, thus illustrating the commutative law.

2. Add $2x+3y+z$ and $-5x+7y+4z$, and check the result by letting $x=3$, $y=2$, and $z=1$.

Add the following, checking each result:

3.

$$\begin{array}{r} 4x+2y-8z \\ 2x+3y-2z \\ \underline{-7x-4y+5z} \end{array}$$

4.

$$\begin{array}{r} -12a-27b+13c \\ -42a-13b-8c \\ \underline{-75a+b-c} \end{array}$$

5.

$$\begin{array}{r} 27.8a^2+52.6b^3-32.9c \\ 35.3a^2-26.4b^3+15.7c \\ \underline{-29.4a^2-0.5b^3-0.4c} \end{array}$$

6.

$$\begin{array}{r} a^3+3a^2b+3ab^2+b^3 \\ a^3-3a^2b+3ab^2-b^3 \\ \underline{a^2b-ab^2} \end{array}$$

Arrange according to the proper powers and add:

7. $x^3+y^3-3x^2y$, $3x^3+3xy^2$, $-4x^3+3x^2y-y^3$.

8. $x^4+y^4+x^3y^2$, x^3y+xy^3, $x^4+2x^3y+7x^2y^2-y^4$.

9. $3(a-b)+7(a-b)^3$, $4(a-b)^2-3(a-b)$, $8(a-b)^3$.

10. $2(a+b)^4-3(a+b)^2+1$, $5(a+b)^3+7(a+b)-2(a+b)^4$.

11. $27(a+b-c)^2-4(a+b-c)+9$, $5(a+b-c)-26(a+b-c)^2-9$.

Simplify the following:

12. $32\sqrt{a+b}-27\sqrt{a-b}+4-3\sqrt{a+b}+6\sqrt{a-b}-8$.

13. $46\sqrt{x^2-xy+y^2}+7\sqrt{x}-5\sqrt{x}+2-45\sqrt{x^2-xy+y^2}$.

14. $a^3-3a^2b-3a^2c+3b^2c+3ac^2-4a^3+3a^2b+3a^2c$.

15. $5x^6-2+3x^4-7-4x^5+x+9-4x^2+3x^5-4x^6+x^5$.

16. $2(a-b)^2+3(a-b)-4+7(a-b)-6(a-b)^2+27(a-b)-42(a-b)^2+3(a-b)+62-7(a-b)^2-32(a-b)$.

Exercise 6. Equations involving Addition

Examples 1 to 9, oral — Examples 10 to 27, written

1. What must be added to both members of the equation $7x - 9 = 21 - 3x$ in order to reduce it to the form $ax = b$?

Reduce to the form $ax = b$:

2. $7x - 12 = 48 - 5x$.

3. $8x - 30 = 60 - 7x$.

4. $6x - 20 = 40 - 6x$.

5. $4x - 34 = 50 - 3x$.

6. $19x + 2x = 42$.

7. $11x - 9 + 2x = 30$.

8. $12x - 1 + 5x = 50$.

9. $20x - 9 + 3x = 60$.

Solve the following, and check the result by substituting it in the original equation:

10. $3.4x = 60 - 8.6x$.

11. $8.7x = 90 - 6.3x$.

12. $5.3x = 84 - 1.7x$.

13. $9.5x = 42 - 10.5x$.

14. $8.9x = 51 - 8.1x$.

15. $19\frac{1}{2}x = 270 - 7\frac{1}{2}x$.

16. $9x - 32.5 = 48.5$.

17. $11x - 42\frac{2}{3} = 34\frac{1}{3}$.

18. $13x - 61 = -9$.

19. $14x - 73\frac{3}{4} = 24\frac{1}{4}$.

20. $16x - 92 = -12$.

21. $21x - 47\frac{5}{8} = 99\frac{3}{8}$.

22. If from 12 times a certain number we subtract 32.8, the result is 27.2. What is the number?

23. If from 17 times a certain number we subtract 26.4, the result is 7.6. What is the number?

24. If from 21 times a certain sum of money we take \$5, there is left \$100. What is the sum of money?

25. If from 32 times a certain distance we take 59.8 ft., there is left 100.2 ft. What is the distance?

26. A man has \$32.25, which is 51 times what he had a week ago, less \$324.75. How much had he a week ago?

27. If from 19.37 times a certain number we take 0.02, the result is 38.72. What is the number?

11. Review of Subtraction. The following laws of subtraction are given in the work in elementary algebra:

1. *To subtract a monomial from a similar monomial, find the difference between the coefficients of the common factor, and multiply this difference by the common factor.*

Thus $9a - 4a = 5a$, $ax - bx = (a - b)x$, and $mx^2 - 4nx^2 = (m - 4n)x^2$. If the monomials are dissimilar, merely indicate the subtraction, as in the case of $4a - 7b$.

2. *To subtract one quantity from another, find a quantity that added to the subtrahend will equal the minuend.*

If the student will think of this when he subtracts, he will rarely be troubled in the matter of signs. If he is confused, he should remember that in taking a smaller number from a larger one the result must be positive, and in taking a larger number from a smaller one the result must be negative.

3. *To subtract one polynomial from another, arrange similar terms under one another and subtract these terms separately.*

For example, subtract $4a^2 - 3ab + 1$ from $6a^2 - 8 - 9ab$. Rearranging, we have the following:

Operation	Check
$6a^2 - 9ab - 8$	$6 - 9 - 8 = -11$
$4a^2 - 3ab + 1$	$4 - 3 + 1 = \quad 2$
$2a^2 - 6ab - 9$	$2 - 6 - 9 = -13$

Here the work is checked by letting $a = 1$ and $b = 1$. If a check upon the exponents is desired, use other values than 1 for a and b. For example, suppose we let $a = 3$ and $b = -4$. We then have $6a^2 - 9ab - 8 = 54 + 108 - 8 = 154$, and $4a^2 - 3ab + 1 = 36 + 36 + 1 = 73$. Furthermore, $2a^2 - 6ab - 9 = 18 + 72 - 9 = 81$, and $154 - 73 = 81$.

Check
$54 + 108 - 8 = 154$
$36 + \quad 36 + 1 = \quad 73$
$18 + \quad 72 - 9 = \quad 81$

4. *To subtract one quantity from another we may mentally change the signs of the subtrahend and proceed as in addition.*

This law is evident from the fact that $a - b = a + (-b)$, and that $x - (-y) = x + (+y)$. It is not necessary to use the law, however, since the others answer the purpose fully.

Exercise 7. Subtraction

1. Draw a line to represent $+2$; one to represent -5; one to represent $2+(-5)$.

Subtract the following, checking each result:

3.

$$\begin{array}{r} 8x^2+5x-7 \\ \underline{2x^2-3x+4} \end{array}$$

4.

$$\begin{array}{r} 5a^2-3ab+2b^2 \\ \underline{7a^2-6ab-9b^2} \end{array}$$

5.

$$\begin{array}{r} 6x^2+3xy-4.6y^2 \\ \underline{8x^2-9xy-7.8y^2} \end{array}$$

6.

$$\begin{array}{r} 0.2m^2+3\sqrt{n}-42.9 \\ \underline{0.7m^2-\sqrt{n}+83.7} \end{array}$$

7.

$$\begin{array}{r} 14.2(a-b)+3.4\sqrt{a+b} \\ \underline{19.9(a-b)-7.9\sqrt{a+b}} \end{array}$$

8.

$$\begin{array}{r} a^3+3a^2b+3ab^2+b^3+4ab \\ \underline{a^3-3a^2b+3ab^2-b^3+4ab} \end{array}$$

9.

$$\begin{array}{r} 7.8x^3-5.2x^2+9.3x-14.1 \\ \underline{0.9x^3-7.1x^2+0.8x-23.5} \end{array}$$

10.

$$\begin{array}{r} a(b-c)+\sqrt{x+y}-\sqrt{m-n} \\ \underline{m(b-c)+n\sqrt{x+y}-\sqrt{m-n}} \end{array}$$

11.

$$\begin{array}{r} a(x+y)+b(y^2+z)+c(z^2+x^2) \\ \underline{b(x+y)+c(y^2+z)+a(z^2+x^2)} \end{array}$$

12.

$$\begin{array}{r} (p+q)\sqrt{x}+x\sqrt{p+q}+m^2-n^2 \\ \underline{(p+q)\sqrt{x}+y\sqrt{p+q}-2m^2-3n^2} \end{array}$$

13. From $0.4x^4-3.2x^3+7.3x^2-0.25x+7.2$ subtract $x^3+2.4x^4-0.25x-2.2+6.3x^2$.

14. From $23.7P+3.04W-F$ subtract the sum of $4.8P-2.4W+3F$ and $8.2P+2.7W-6F$.

15. From $27\sqrt{x^2+x+1}-4\sqrt{x^2-x+1}+9.87$ subtract the sum of $5.4\sqrt{x^2+x+1}-8.92$, 152.93, $6\sqrt{x^2-x+1}$, and $4.98+\sqrt{x^2-x+1}$.

16. From the sum of $4.2x^3-2.8x^2-4.2x+7$ and $5.8x^3-7.2x^2+5.8x-3$ subtract the sum of $x^3-7.3x^2+x^2+2.92$ and $9x^3-2.7x^2+0.6x^2+1.08$.

Exercise 8. Equations involving Subtraction

Solve these equations and check the results:

1. $9x - x = 16$.
2. $2x + \frac{3}{4} = 2\frac{3}{4}$.
3. $7x - x = 24$.
4. $7x - 2x = 15$.
5. $8x - 5x = 45$.
6. $8x - 7x = 50$.
7. $6x - 3x = 75$.
8. $6x - 3x = 78$.
9. $5x - 3x = 50$.
10. $6x + \frac{3}{4} = 6\frac{1}{4}$.
11. $6x + 9 = 64$.
12. $7x + 8 = 57$.
13. $7x + 8 = 58$.
14. $8x + 7 = 71$.
15. $8x + 7 = 75$.
16. $9x + 6 = 60$.
17. $9x + 6 = 63$.
18. $9x + 6 = 66$.
19. $15a - 3a = 60$.
20. $15a - 3a = 66$.
21. $17a - 4a = 26$.
22. $17a - 4a = 27$.
23. $21y - 7y = 28$.
24. $21y - 7y = 35$.
25. $33m - 4m = 29$.
26. $33m - 4m = 32$.
27. $32p - 7p = 80$.

Solve these equations:

28. $9x + 1 = 2x + 15$.
29. $9x + 3 = 3x + 27$.
30. $9x + 5 = 5x + 25$.
31. $1.2x + 4 = 0.9x + 16$.
32. $1.4x + 3 = 0.8x + 39$.
33. $1.5x + 0.8 = 0.9x + 6.8$.

34. If we add 3 to 12 times a certain number, the result is 123. What is the number?

35. If we add 5 to 8 times a certain number, the result is 29. What is the number?

36. If we add 7 to 6 times a certain number, the result is 43. What is the number?

37. If from 6 times a certain number we take twice the number, the result is 48. What is the number?

38. If from 10 times a certain number we take 7 times the number, the result is 39. What is the number?

39. If 7 times your age plus 4 times your age is 165 years, how old are you?

40. If from 9.8 times a certain number we take 6.9 times the number, the result is 96. What is the number? Check the result by substituting in the problem.

12. Removal of Parentheses. If we subtract $b + c$ from a, and $b - c$ from a, we have the following results:

$$\begin{array}{r} a \\ b+c \\ \hline a-b-c \end{array} \qquad\qquad \begin{array}{r} a \\ b-c \\ \hline a-b+c \end{array}$$

We therefore see that

$$a-(b+c)=a-b-c,$$

and

$$a-(b-c)=a-b+c.$$

If an expression inclosed within parentheses is preceded by the negative sign, the parentheses may be removed, provided the sign before each term is changed.

If an expression inclosed within parentheses is preceded by the positive sign, the parentheses may be removed without any change of sign.

That is, $x+y-(x-y)=x+y-x+y=2y;$

and $$\frac{3}{2}-\frac{5-4}{2}=\frac{3-5+4}{2}=\frac{2}{2}=1,$$

the fraction bar having the force of parentheses.

Exercise 9. Removal of Parentheses

Remove the parentheses and simplify the results:

1. $7.2\,x-(3.4\,x+5)$.
2. $8.5\,m+(4.5\,m-7)$.
3. $2.3\,p-(2.3\,p-q)$.
4. $54\,x^2+(46\,x^2-100)$.
5. $29\,x^2y^2+(x^2y^2-3)$.
6. $31\,p^2-(42\,p^2-81)$.
7. $27\,mn-(52\,mn+7)$.
8. $(63\,x+y)-3\,x-y$.
9. $41\,ab-(27\,ab+b^2)$.
10. $6.8\,x^2y^2+(7.2\,x^2y^2+3)$.
11. $54.8\,x^4-(75.2\,x^4+9)$.
12. $85.6\,p^2-(29.4\,p^2+1)$.
13. $0.8\,x^2y^2+(0.2\,x^2y^2+xy)$.
14. $(5\,x^2-7\,xy+3\,y^2)-3\,y^2$.
15. $-(2\,x+3\,y)+(3\,x+2\,y)$.
16. $-(7\,ab+2)-(8\,ab-2)$.
17. $a^3+7\,a^2b-8\,ab^2-b^3-(2\,a^3-9\,a^2b-6\,ab^2+9\,b^3)$.

13. Removal of Several Symbols of Aggregation. The symbols of aggregation most frequently used in algebra are the following:

Parentheses, as in $a-(b+c)$; *Brackets*, as in $p-[q-r]$;
Bar, or *vinculum*, as in $x-\overline{y-z}$; *Braces*, as in $m-\{m+n\}$.

When one symbol of aggregation incloses another, we may remove either the outer one or the inner one first. Beginners usually find it less confusing to remove the inner one first.

1. Required to simplify the expression $a-(b-\overline{c-d})$.

$$\begin{aligned} a-(b-\overline{c-d}) &= a-(b-c+d) \\ &= a-b+c-d. \end{aligned}$$

2. Required to simplify the expression $3.8\,a-[10\,a-(a-b)]$.

$$\begin{aligned} 3.8\,a-[10\,a-(a-b)] &= 3.8\,a-[10\,a-a+b] \\ &= 3.8\,a-[9\,a+b] \\ &= 3.8\,a-9\,a-b \\ &= -5.2\,a-b. \end{aligned}$$

3. Required to simplify the expression

$$20\,a-\{10\,a-[6\,a-\overline{c-(5\,a-b)}]+c\}.$$

$$\begin{aligned} & 20\,a-\{10\,a-[6\,a-\overline{c-(5\,a-b)}]+c\} \\ = {} & 20\,a-\{10\,a-[6\,a-\overline{c-5\,a+b}]+c\} \\ = {} & 20\,a-\{10\,a-[6\,a-c+5\,a-b]+c\} \\ = {} & 20\,a-\{10\,a-6\,a+c-5\,a+b+c\} \\ = {} & 20\,a-10\,a+6\,a-c+5\,a-b-c \\ = {} & 21\,a-b-2\,c. \end{aligned}$$

Problems like Ex. 3 are so rarely found in algebra that they may be omitted unless the student is preparing for some examination in which they are required.

14. Insertion of Parentheses. From what we have learned of the removal of parentheses we see that

Two or more terms may be inclosed in parentheses preceded by a plus sign without changing the signs of the terms.

Two or more terms may be inclosed in parentheses preceded by a minus sign, provided the sign of each of the terms is changed.

Exercise 10. Removal of Parentheses

Remove the symbols of aggregation and simplify:

1. $12a-(7a-b)+(6a-4b)$.
2. $17xy-(15xy+4)-(2xy-8)$.
3. $25ab-(5ab+c)+(5ab-c)$.
4. $(a-b)-(b-c)-(c-d)-(d-e)-(e-a)$.
5. $a-(b-c)+b-(c-d)+c-(d-a)$.
6. $a^2+2ab+b^2-(a^2-2ab+b^2)$.
7. $a^2-(2ab-b^2)-[a^2-(2ab+b^2)]$.
8. $x^2-[x^2-(x^2-2xy+y^2)+2xy-y^2]$.
9. $a^3-(3a^2b-3ab^2)+b^3-[a^3-(3a^2b-3ab^2+b^3)]$.
10. $a+7-(2a-7)-[a-7-(2a-\overline{3a-7})]$.
11. $x^3-(3x^2y-3xy^2)+[y^3-x^3+(3x^2y-3xy^2+y^3)]$.
12. $a-[a+b-(c-d+e-a)+c]-b-c+d-e$.
13. $2a-\{3a+b-c-4c+[3a-(b-\overline{c-2b})]\}$.
14. $7a-\{2a-b+c+4c-[4a+(a-b+3c)]\}$.
15. $25ab-\{17ab-(ab+1)+[ab-(ab+\overline{ab-1})]\}$.
16. $[a^2+2ab+b^2]-[a^2-2ab+b^2]-[-a^2+2ab-b^2]$.
17. $15a^2b-\{15ab^2+12-[15a^2b-12+(15ab^2-12)]\}$.

Remove the parentheses, leaving the brackets:

18. $[a^2-(b^2+c^2)]\times[a^2-(b^2-c^2)]$.
19. $[ab-(cd+1)]\times[ab-(cd-1)]$.
20. $[x^2-(2xy-y^2)]\times[x^2+(2xy+y^2)]$.
21. $[(a-b)-(b-a)]\times[(a+b)-(b+2a)]$.

22. In the expression $a^2-2ab+b^2$ inclose the last two terms in parentheses, arranging the signs so as not to change the value of the expression.

23. In the expression $a^2+2ab+b^2$ inclose the last two terms in parentheses without changing the value.

15. Laws of Multiplication. The following laws of multiplication are given in elementary algebra:

1. *In multiplying monomials, the exponent of any letter in the product is equal to the sum of the exponents of that letter in the factors.*

That is, $$a^m \cdot a^n = a^{m+n}.$$

This is apparent because if a is taken m times as a factor, and also n times as a factor, it is taken in all $m + n$ times as a factor. In particular,

$$a^3 = aaa,$$

and $$a^5 = aaaaa,$$

whence $$a^3 \cdot a^5 = aaaaaaaa = a^8.$$

2. *A power of a power of a number is equal to the power of the number indicated by the product of the exponents.*

That is, $$(a^m)^n = a^{mn}.$$

For example, $(a^3)^2 = (aaa)^2 = aaaaaa = a^6$. The same reasoning evidently holds, whatever exponents are taken.

3. *In multiplication, two like signs produce plus, two unlike signs produce minus.*

That is, $$+a \cdot (+b) = +ab, \quad +a \cdot (-b) = -ab,$$
$$-a \cdot (-b) = +ab, \quad -a \cdot (+b) = -ab.$$

16. Arranging a Polynomial. A polynomial in which the exponents of a certain letter in the successive terms decrease from left to right is said to be *arranged according to the descending powers* of that letter.

Thus $x^3 + 3x^2 - 4x + 1$ is arranged according to the descending powers of x, and $an^2 + a^2n - a^3$ is arranged according to the descending powers of n.

Similarly, a polynomial may be arranged according to the *ascending* powers of a letter. Thus $a^3 - 3a^2b + 3ab^2 - b^3$ is arranged according to the ascending powers of b.

In multiplying, it simplifies the work if the polynomials are arranged according to the ascending or descending powers of some letter.

17. Illustrative Problems. The work in multiplication may be reviewed by considering these examples:

1. Multiply $-3x^2+2x^4-5x+3x^3-8$ by x^2-x+9.

OPERATION	CHECK
Rearranging, we multiply thus:	
$2x^4+3x^3-3x^2-5x-8$	$=-11$
x^2-x+9	$=9$
$2x^6+3x^5-3x^4-5x^3-8x^2$	-99
$-2x^5-3x^4+3x^3+5x^2+8x$	
$18x^4+27x^3-27x^2-45x-72$	
$2x^6+x^5+12x^4+25x^3-30x^2-37x-72$	$=-99$

In the check we have let $x=1$. Any other value may be taken.

2. Multiply $x^2-3xy+3y^2$ by y^2+x^2.

Rearranging, we multiply thus:

$x^2-3xy+3y^2$	$=1$
x^2+y^2	$=2$
$x^4-3x^3y+3x^2y^2$	2
$x^2y^2-3xy^3+3y^4$	
$x^4-3x^3y+4x^2y^2-3xy^3+3y^4$	$=2$

18. Detached Coefficients. Where we can easily tell what letters will enter in the product, and in what order, we may detach the coefficients in multiplying, thus saving the trouble of writing all the letters. Thus in the first example in § 17 we have

$2+3-3-5-8$	$=-11$
$1-1+9$	$=9$
$2+3-3-5-8$	-99
$-2-3+3+5+8$	
$18+27-27-45-72$	
$2+1+12+25-30-37-72$	$=-99$

We may then write in the letters, and we have

$$2x^6+x^5+12x^4+25x^3-30x^2-37x-72.$$

Exercise 11. Multiplication

Multiply and check:

1. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.
2. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
3. $x^3 - 4x^2 + 22x - 128$ by $x^2 - 2x + 30$.
4. $m^3 - 2.2m^2 + 3.4m - 2.8$ by $3m^2 - 10m + 5$.
5. $x^4 + x^3 - x^2 - x + 1$ by $x^2 - 3x + 40$.
6. $2x^3 - 4x^2 + 8x - 7$ by $4x^2 - 3x + 2$.
7. $5x + 3x^3 - 2 + 7x^2$ by $27 + 4x^2 - 7x$.
8. $24xy + 32x^2 - 15y^2 + 1$ by $xy - y^2 + x^2$.
9. $53p^2q^2 - 41p^3 + q^3$ by $24p^3 + 13q^2 - p^2q^2$.
10. $a^3 + b^3 - a^2b - ab^2$ by $b^3 - a^3 - ab^2 + 4a^2b$.
11. $a^3x^3 + 7ax - 2a^2x^2 - 9$ by $4ax - 2 + a^2x^2$.
12. $a(x + y)^2 - b(x + y) + c$ by $b(x + y)^2 - a(x + y) - c$.
13. $p^4x^4 + 1 - p^3x^3 - p^2x^2 - px$ by $p^3x^3 - px + 3$.
14. $25x^2 + 50xy + 1$ by $25x^2 - 50xy + 1$.
15. $a^2b^2x^2 - 9abx + 4$ by $a^2b^2x^2 + 9abx + 4$.
16. $a^n + a^{n-1} + a^{n-2}$ by a^2; also by a^3; also by $a^2 + 1$.
17. $a^{2m} + b^{2m}$ by $a^{2m} - b^{2m}$, and this product by $a^{4m} + b^{4m}$.
18. $p^m - p^{m-1} + p^{m-2} - p^{m-3}$ by $p^2 + 1$.
19. $a^{m+1} + a^m + a^{m-1} + a^{m-2}$ by $a + 1$.
20. $x^{2m} - 2x^m b^m + b^{2m}$ by $x^{2m} + 2x^m b^m + b^{2m}$.

Detach the coefficients and multiply, writing the product in full:

21. $a^3 + 3a^2 - 4a + 1$ by $a^2 - 2a + 3$.
22. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x^2 + 2xy + y^2$.
23. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^3 - 3x^2y + 3xy^2 - y^3$.
24. $a^3b^3 - 3a^2b^2 + 4ab - 5$ by $a^3b^3 - a^2b^2 + 4ab - 3$.
25. $a^2b^2c^2 - 2abcd + d^2$ by $a^2b^2c^2 + 2abcd + d^2$.

Exercise 12. Equations involving Multiplication

Examples 1 to 3, oral — Examples 4 to 27, written

1. If $x^2 + 7 = x^2 + x$, what should be subtracted from both members to obtain the value of x? What does x equal?

2. If $x^2 + 7 = x(x+1)$, what is the first step in the solution? the next step? What does x equal?

3. If $(x-7)(x+2) = x(x+9)$, what is the first step in the solution? the next step? How will you check the result?

Solve the following equations, checking the result by substituting it in the original equation:

4. $x^2 + 9 = x(x+3)$.

5. $x^2 + 95 = x(x+5)$.

6. $x^2 + 35 = x(x+7)$.

7. $x^2 - 36 = x(x-9)$.

8. $x^2 - 42 = x(x-6)$.

9. $x^2 - 42 = x(x-2)$.

10. $3x(x+1) = 3x^2 + 9$.

11. $5x(x+2) = 5x^2 + 90$.

12. $7x(x+4) = 7x^2 - 28$.

13. $6x(x-3) = 6x^2 - 18$.

14. $4x(x-7) = 4x^2 - 56$.

15. $x(2x-4) = 2x^2 + 40$.

16. If $(n^2 + 2n + 1)(n^2 - 2n + 1) = n^4 - 2n^2 + n$, what is the value of n? Check by substituting in the original equation.

Solve the following equations, checking as before:

17. $(n+1)(n+3) = n(n+2) + 4$.

18. $(a+5)(a+3) = a(a+4) + 7$.

19. $(p+5)(p-3) = p^2 + p + 12$.

20. $(q+7)(q+5) = 2(4q+5) + q^2$.

21. $(x^2 + 2x + 1)(x+3) = x^2(x+5) + 6(x+1)$.

22. $(x^2 - x - 1)(2x+4) = 2x^2(x+1) - 7(x-1)$.

23. $(x^2 - 3x - 7)(x-1) = x^2(x-4) - 5(x-2)$.

24. $(x^2 - 2x - 6)(x+1) = x^2(x-1) - 9(x-1)$.

25. $(x^2 + 4x + 4)(x+1) = x^2(x+5) + 7(x+2)$.

26. $(x^2 - x + 1)(x+1) + x = (x+2)(x^2 - 2x + 4)$.

27. $(x^2 + x + 1)(x-1) + 3x = (x+2)(x^2 - 2x + 4)$.

19. Laws of Division. The following laws of division are given in elementary algebra:

1. *The exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus the exponent of that letter in the divisor.*

That is, $a^m \div a^n = a^{m-n}$.

For if we have $a^7 \div a^3$, we may indicate the work as follows:

$$\frac{a^7}{a^3} = \frac{aaaa\cancel{aaa}}{\cancel{aaa}} = a^4.$$

That is, from the seven equal factors in the dividend we have removed three of these factors, and four factors are left. Similarly, if from m equal factors we remove n of these factors, $m - n$ of them are left.

The case in which m is less than n is considered later in the work.

2. *In division, two like signs produce plus; two unlike signs produce minus.*

That is, $\frac{4}{2} = 2$, $\frac{-4}{-2} = 2$, $\frac{-4}{2} = -2$, and $\frac{4}{-2} = -2$.

20. Illustrative Problem. Required to divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.

OPERATION

$$\begin{array}{l|l} a^3 - 3a^2b + 3ab^2 - b^3 & a^2 - 2ab + b^2 \\ \underline{a^3 - 2a^2b + \ \ ab^2} & \underline{a - b} \\ \quad - a^2b + 2ab^2 - b^3 & \\ \quad \underline{- a^2b + 2ab^2 - b^3} & \end{array}$$

CHECK

Let $a = 2, b = 1$.

Then $\frac{1}{1} = 1$.

In this check we cannot let $a = b$, because this would make the divisor zero, and zero divisors are excluded from division. Letting $a = 2$ and $b = 1$, the dividend is 1, the divisor 1, and the quotient 1.

Since we cannot divide by zero, therefore in checking the work we use some value for the letters that shall not make the divisor zero.

It is also a good check to notice that if both dividend and divisor are homogeneous, the quotient is also homogeneous. In the case given above the dividend, divisor, and quotient are all homogeneous.

21. Detached Coefficients. Where we can easily tell what letters will enter in the quotient, and in what order they will appear, we may detach the coefficients in dividing, thus saving the trouble of writing all the letters. For example, consider the following cases:

1. Divide $x^2 - 5x - 84$ by $x - 12$.

OPERATION

$$\begin{array}{r|l} 1 - 5 - 84 & 1 - 12 \\ \underline{1 - 12} & \overline{1 + 7} \\ 7 - 84 & \\ \underline{7 - 84} & \end{array}$$

CHECK

$$\frac{-88}{-11} = 8.$$

Inserting the letters, the quotient is $x + 7$.

In the check it is convenient to let $x = 1$, since we have only to add the coefficients in dividend, divisor, and quotient.

2. Divide $x^3 - y^3$ by $x - y$.

Since an arrangement is desirable according to descending powers of x and ascending powers of y, with all the powers shown, we may think of $x^3 - y^3$ as $x^3 + 0x^2y + 0xy^2 - y^3$.

OPERATION

$$\begin{array}{r|l} 1 + 0 + 0 - 1 & 1 - 1 \\ \underline{1 - 1} & \overline{1 + 1 + 1} \\ 1 + 0 & \\ \underline{1 - 1} & \\ 1 - 1 & \\ \underline{1 - 1} & \end{array}$$

CHECK

Let $x = 2, y = 1$.

Then $7 = 4 + 2 + 1$.

Inserting the letters, the quotient is $x^2 + xy + y^2$.

Here we cannot let $x = y = 1$ in the check, because this would make the divisor zero. Letting $x = 2$ and $y = 1$, the dividend is 7, the divisor 1, and the quotient 7.

If there is a remainder in division, a fraction is written in the quotient, as in the case of numbers. For example, $(x^2 + y^2) \div (x + y) = x - y + \dfrac{2y^2}{x + y}$. To check this result, let $x = 3$ and $y = 2$. We then have $13 \div 5 = 2\frac{3}{5} = 1\frac{8}{5}$.

Exercise 13. Division

Divide and check:

1. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
2. $p^3 - 3p^2q + 3pq^2 - q^3$ by $p - q$.
3. $m^4 + 4m^3n + 6m^2n^2 + 4mn^3 + n^4$ by $m + n$.
4. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x - y$.
5. $a^3 + 5a^2b + 8ab^2 + 4b^3$ by $a + b$.
6. $x^4 + 3x^3 - 2x^2 - 12x - 8$ by $x + 1$.
7. $x^5 + 4x^4 + x^3 - 14x^2 - 20x - 8$ by $x + 1$.
8. $p^5 + 9p^4 + 21p^3 + 20p^2 + 12p$ by $p + 1$.
9. $x^3 - 13x^2 + 40x - 36$ by $x^2 - 4x + 4$.
10. $m^4 - 11m^3 + 5m^2 + 47m + 30$ by $m - 3$.
11. $x^4 - 11x^3y + 5x^2y^2 + 47xy^3 + 30y^4$ by $x^2 + 2xy + y^2$.
12. $m^3 + 12m^2n + 39mn^2 + 28n^3$ by $m^2 + 5mn + 4n^2$.
13. $a^4 - 16b^4$ by $a + 2b$; by $a - 2b$; by $a^2 + 4b^2$.
14. $32x^5 + 243y^5$ by $2x + 3y$; by $2x + y$; by $2x - y$.
15. $x^4 - 16x + x^3 - 4 - 9x^2$ by $4x + 4 + x^2$.
16. $6pq - p^2q^2 + p^4q^4 - 9$ by $p^2q^2 + 3 - pq$.
17. $a^3 + b^3$ by $a + b$; by $a - b$; by $a^2 - ab + b^2$; by $a^2 + b^2$.
18. $x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$ by $x + 1$.
19. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ by $a^2 + 2ab + b^2$.
20. $(a + b)^3 + 3(a + b)^2 + 3(a + b) + 1$ by $(a + b)^2 + 2(a + b) + 1$.

Detach the coefficients and divide, writing the quotient in full:

21. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.
22. $m^3 - 3m^2n + 3mn^2 - n^3$ by $m^2 - 2mn + n^2$.
23. $x^3 - y^3$ by $x + y$; by $x - y$; by $x^2 + y^2$; by $x^2 - y^2$.
24. $x^4 + 2x^3 + 3x^2 + 2x + 1$ by $x^2 + x + 1$.
25. $6x^4 + 5x^3 - 4x^2 + 55x - 14$ by $2x^2 - 3x + 7$.
26. $x^5 - y^5$ by $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

Exercise 14. Equations involving Division

Examples 1 to 11, oral — Examples 12 to 32, written

1. If $7x = 84$, what is the value of x? How do you prove it?
2. If $ax = 9a$, what is the value of x? Prove it.
3. If $2a^2x = 8a^4$, what is the value of x? Prove it.

Find the value of x and prove that it is correct:

4. $3ax = 12a$.
5. $2a^2x = 10a^2$.
6. $7abx = 28a^2b^2$.
7. $9abx = 81a^3b^3$.
8. $-2ax = 36a^2$.
9. $4ax = -36a^3$.
10. $-7a^nx = -84a^n$.
11. $-abcx = -a^2b^2c^2$.
12. If $(a+b)x = a^2 + 2ab + b^2$, what is the value of x?

Find the value of x and prove that it is correct:

13. $bx = a^3 + a^2b$.
14. $a^5x = a^9 - 3a^5$.
15. $a^7x = 14a^{17} - a^7$.
16. $-ax = -14a^2 + a$.
17. $-8\frac{1}{2}ax = -357a^m$.
18. $(a-b)x = a^3 - b^3$.
19. $(a+b)x = 3(a^3 + b^3)$.
20. $(a-7)x = a^3 - 343$.
21. $(a+7)x = a^3 + 343$.
22. $(a^2 - b^2)x = a^4 - b^4$.

Given the following polynomials:

$$M = a^2 + 2ab + b^2, \qquad P = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$N = a^2 - 2ab + b^2, \qquad Q = a^3 - 3a^2b + 3ab^2 - b^3,$$

find the value of x in the following:

23. $Mx = 5P$.
24. $Mx = -7P$.
25. $Mx = -11P$.
26. $Nx = 2Q$.
27. $Nx = -Q$.
28. $Nx = -11Q$.
29. $(a-b)x = 2Q$.
30. $(a+b)x = 4P$.
31. $MNx = 4PQ$.

32. If $i = prt$, what is the value of t in terms of i, p, and r? Evaluate the result if $i = 54$, $p = 900$, and $r = 6\%$; if $i = 324$, $p = 900$, and $r = 6\%$.

22. Synthetic Multiplication. There is an abbreviation of multiplication that is convenient when the method of detached coefficients can be employed, and when the first coefficient of the multiplier is 1. This is known as *synthetic multiplication.*

In multiplying $2x^2-3x+7$ by x^2+2x-3, using detached coefficients, we have the following:

$$\begin{array}{ll} 2-3+7 & =6 \\ 1+2-3 & =0 \\ \hline 2-3+7 & 0 \\ \quad 4-6+14 & \\ \quad\quad -6+\ 9-21 & \\ \hline 2+1-5+23-21 & =0 \end{array}$$

We see that all the essential figures of this work appear in the following condensed form:

$$\begin{array}{r|l} & 2-3+7 \\ +2 & \quad 4-6+14 \\ -3 & \quad\quad -6+\ 9-21 \\ \hline & 2+1-5+23-21 \end{array}$$

The product in each case is $2x^4+x^3-5x^2+23x-21$.

Since the first multiplier is 1, we may use the $2-3+7$ of the multiplicand, just as it stands, for the first partial product. We have therefore saved writing four figures, and our work is considerably more compact.

In multiplying $5a^2+3ab-7b^2$ by $a-3b$ we have

$$\begin{array}{r|ll} & 5+\ 3-\ 7 & =\ 1 \\ -3 & \ -15-\ 9+21 & =-2 \\ \hline & 5-12-16+21 & =-2 \end{array}$$

The product is $5a^3-12a^2b-16ab^2+21b^3$.

If the student will perform both of the above multiplications without detaching the coefficients, and by the long method that he first learned, he will see the gain that he makes by synthetic multiplication. The gain is still more noticeable in a case in which the multiplicand and multiplier have a larger number of terms.

23. Synthetic Division. There is an abbreviation of division that is convenient when the divisor is a binomial whose first coefficient is 1, and when the method of detached coefficients can be employed. This is known as *synthetic division.*

In dividing $2a^3 - 9a^2 + 14a - 18$ by $a - 3$, using detached coefficients, we have the following:

$$\begin{array}{r|l} 2 - 9 + 14 - 18 & 1 - 3 \\ \underline{2 - 6} & \underline{\quad\quad} \\ \quad - 3 + 14 & 2 - 3 + 5 \\ \quad \underline{- 3 + 9} & \\ \qquad\quad 5 - 18 & \\ \qquad\quad \underline{5 - 15} & \\ \qquad\quad\quad - 3, \text{ Remainder} & \end{array}$$

We see that all the essential figures of this work appear in the following condensed form:

$$\begin{array}{r|l} & 2 - 9 + 14 \quad - 18 \\ 3 & \underline{\quad + 6 - \quad 9 \quad\quad 15} \\ & 2 - 3 + \quad 5, - \quad 3 \end{array}$$

The method of obtaining these figures is very simple. The first figure, 2, is brought down as the first figure of the quotient. This is multiplied by 3, which is the -3 in $a - 3$, and which is written at the left. The result, $+6$, is written under -9 of the dividend. Then $-9 + 6 = -3$, and this is written below the line as the second quotient figure. Then $3 \times (-3) = -9$, and this is added to 14, the sum, 5, being written below. Then $3 \times 5 = 15$, and this is added to -18, giving -3, the figure of the remainder.

By comparing this with the work above it will be seen that the essential figures are exactly the same, except that by changing the sign of -3 we are enabled to add each time instead of subtracting.

In dividing $x^3 - 9x^2 + 15x - 2$ by $x - 2$ we have

$$\begin{array}{r|l} & 1 - 9 + 15 \quad - 2 \\ 2 & \underline{\quad\quad 2 - 14 \quad\quad 2} \\ & 1 - 7 + \quad 1, \quad 0 \end{array}$$

Here the quotient is $x^2 - 7x + 1$, and the remainder is 0. This is, therefore, a convenient method of determining factors.

Exercise 15. Multiplication and Division

Multiply, using the synthetic method:

1. $(4x+9)(x-3)$.
2. $(5a+17)(a+4)$.
3. $(7p+32)(p-5)$.
4. $(8m+23)(m+7)$.
5. $(51y-15)(y-3)$.
6. $(62x+3)(x-9)$.
7. $(75x+7)(x+8)$.
8. $(27P+4)(P+12)$.
9. $(41W+3)(W-7)$.
10. $(52F+7)(F+3)$.
11. x^3+3x^2-4x+7 by $x-3$.
12. $x^4-3x^3+5x^2-4x+9$ by $x+8$.
13. $x^4+2x^3y-3x^2y^2+5xy^3+3y^4$ by $x-3y$.
14. $a^5+4a^4-3a^3+5a^2+7a-3$ by $a+5$.
15. a^3+3a^2+4a-5 by a^2+2a-3.
16. $x^4+4x^3-3x^2+5x-7$ by x^2-3x+5.
17. $x^4-3x^3y+4x^2y^2-5xy^3+2y^4$ by $x^2+2xy-3y^2$.
18. $a^5+3a^4-2a^3+4a^2-5a+7$ by a^2-3a-5.

Divide, using the synthetic method:

19. $x^2+8x+15$ by $x+3$.
20. $x^2+10x+24$ by $x+4$.
21. $x^2+7x-18$ by $x-2$.
22. $3x^2+x-4$ by $x-1$.
23. $4x^2-3x-10$ by $x-2$.
24. $4x^2+21x+20$ by $x+4$.
25. $x^2+11x+28$ by $x+7$.
26. $x^2+5x-28$ by $x-7$.
27. $a^2+3a-28$ by $a+7$.
28. $a^2-4a-45$ by $a-9$.
29. P^2-49 by $P-7$.
30. P^2-52 by $P-7$.
31. $x^3-3x^2y+3xy^2-y^3$ by $x-y$.
32. $x^3+3x^2y+3xy^2+y^3$ by $x-y$.
33. $x^4+x^3-x^2-3x-14$ by $x-2$.
34. $x^5+4x^4-3x^3+2x^2-7x+1$ by $x-5$.
35. $x^5+3x^4-12x^3+6x^2-4x+7$ by $x-3$.
36. $x^7+4x^5+3x^3-2x^2+3x+1$ by $x-1$.
37. $x^6+3x^4+2x^3-3x^2+5x+6$ by $x-2$.

Multiply as indicated:

38. $(3a+7)(3a+19)$.

39. $(3a+7)(3a-19)$.

40. $(8p+q)(8p-4q)$.

41. $(9p+4q)(9p+7q)$.

42. $(5m+3n)(5m-11n)$.

43. $(7m+4n)(7m-9n)$.

44. $(6a+3b)(6a-7b)$.

45. $(7a-3b)(7a+9b)$.

46. $(a^n+4)(a^n-17)$.

47. $(5a^m+16)(5a^m-27)$.

48. $(a^mb^n+4)(a^mb^n+7)$.

49. $(a^mb^n+4)(a^mb^n-15)$.

50. $(2x-3y-z)(2x-3y+z)$.

51. $(7x-5y+3z)(5y+3z+7x)$.

52. $(3x^2-7+2y^2)(7+3x^2+2y^2)$.

53. $(7ab-4-7c^2)(7c^2+4+7ab)$.

54. $(5x+3y-\frac{1}{2})(5x-3y+\frac{1}{2})$.

55. $(9x^2-7xy+y^2)(y^2+9x^2+7xy)$.

56. $(a^2-5b^2-12c^2)(a^2-5b^2+12c^2)$.

57. $(15x^2-13y^2+3z^2)(15x^2+13y^2+3z^2)$.

58. Multiply $a+x+y$ by $a-x-y$; by $a-x+y$; by $a+x-y$; by $-a-x-y$.

59. Multiply $a^2+2ab+b^2$ by $a+b$; by $a^2-2ab+b^2$; by $(a+b)^2$; by $a^2+2ab-b^2$.

60. Multiply $4a^2-12ab+9b^2$ by $2a-3b$, and $2a+3b$ by $4a^2+12ab+9b^2$.

61. Multiply $x+13$ by $x+7$; by $x+11$; by $x+13$; by $x+21$; by $x+17$; by $x-13$; by $x-17$.

62. Multiply $2x+11$ by $2x-11$; by $2x+11$; by $2x+13$; by $2x-13$; by $2x+15$; by $2x-15$.

63. Expand $(a-7)^2$; $(5a-19)^2$; $(17x+11)^2$; $(x^m+1)^2$.

64. Expand $(x-2)^3$; $(2x+7)^3$; $(11x-5)^3$; $(x^{2m}-3)^3$.

65. Expand $(a^2+a+1)^2$; $(a^2-a+1)^2$; $(a^2+a-1)^2$.

66. Expand $(2x^2y^2+3xy+1)^2$; $(3x^2y^2-4x+7)^2$.

67. Expand $(x^2+2x+1)^2$; $(x^2+2x+1)^3$; $(x^2-2x+1)^3$.

24. Special Products. The following special products have been studied in elementary algebra:

1. *The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.*

That is, $(a+b)^2 = a^2 + 2ab + b^2$.

Therefore

$12^2 = (10+2)^2 = 10^2 + 2 \times 10 \times 2 + 2^2 = 144,$

$35^2 = (30+5)^2 = 900 + 2 \times 30 \times 5 + 25 = 1225.$

ab	b^2
a^2	ab
a	b

It is easily seen that the figure representing the square on $a+b$ is made up of a^2, ab, ab, and b^2, or $a^2 + 2ab + b^2$.

2. *The square of the difference of two numbers is the square of the first, minus twice their product, plus the square of the second.*

That is, $(a-b)^2 = a^2 - 2ab + b^2$.

Therefore $(2x - \frac{1}{4})^2 = 4x^2 - 2 \cdot 2 \cdot \frac{1}{4}x + \frac{1}{16}$

$= 4x^2 - x + \frac{1}{4}.$

3. *The product of the sum and the difference of two numbers is the difference of their squares.*

That is, $(a+b)(a-b) = a^2 - b^2$.

Therefore $(3m+4)(3m-4) = 9m^2 - 16.$

4. *The product of two binomials having a common term equals the square of the common term, plus the product of the common term by the sum of the other terms, plus the product of the other terms.*

That is, $(x+a)(x+b) = x^2 + (a+b)x + ab$.

Therefore $(x+7)(x-9) = x^2 - 2x - 63.$

5. *The cube of the sum of two numbers is the cube of the first, plus three times the square of the first multiplied by the second, plus three times the first multiplied by the square of the second, plus the cube of the second.*

That is, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$

and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

Therefore $(a+2)^3 = a^3 + 3a^2 \cdot 2 + 3a \cdot 2^2 + 2^3$

$= a^3 + 6a^2 + 12a + 8.$

Exercise 16. Special Products

Multiply or raise to powers as indicated:

1. $(a+14)^2$.
2. $(a-13)^2$.
3. $(2x+11)^2$.
4. $(3x-13)^2$.
5. $(5p+2q)^2$.
6. $(7p-3q)^2$.
7. $(8x^2+15)^2$.
8. $(9x^3-11)^2$.
9. $(7x^2+2y^2)^2$.
10. $(9x^3-2y^2)^2$.
11. $(11x^3+3)^2$.
12. $(12x^3-5)^2$.
13. $(14x^2y^2+3)^2$.
14. $(15x^2y^2-2)^2$.
15. $(7+3xyz)^2$.
16. $(5-7xyz)^2$.
17. $(ab+3c)^2$.
18. $(3ab-c)^2$.
19. $(4xy+5)^2$.
20. $(5xy-4)^2$.
21. $(a^3b^2c+1)^2$.
22. $(1-a^3b^2c)^2$.
23. $(5pqr+1)^2$.
24. $(5pq-7r)^2$.
25. $(4xyz+3)^2$.
26. $(3xyz-4)^2$.
27. $(7p^2q^2r^2+11)^2$.
28. $(9ab^2c^3d^4+2)^2$.
29. $(a+2.9b)(a-2.9b)$.
30. $(2.4a+1)(2.4a-1)$.
31. $(1.3m+2)(1.3m-2)$.
32. $(0.4x+y)(0.4x-y)$.
33. $(0.5x+0.2y)(0.5x-0.2y)$.
34. $(7.5pq+1.7)(7.5pq-1.7)$.
35. $(6.3a^2+b)(6.3a^2-b)$.
36. $(5.2x^3+1)(5.2x^3-1)$.
37. $(4\frac{1}{2}p^2+1)(4\frac{1}{2}p^2-1)$.
38. $(2\frac{2}{3}a^4+3)(2\frac{2}{3}a^4-3)$.
39. $(3\frac{4}{5}m^5+2)(3\frac{4}{5}m^5-2)$.
40. $(P^7+14W)(P^7-14W)$.
41. $(E^6+9.1F)(E^6-9.1F)$.
42. $(x^3y^2z-8.1)(x^3y^2z+8.1)$.
43. $(m^4n^4-7.4)(m^4n^4+7.4)$.
44. $(8.3+P^4Q^4)(8.3-P^4Q^4)$.
45. $(x+17)(x-19)$.
46. $(x-23)(x+42)$.
47. $(xy+15)(xy+18)$.
48. $(x^2y-14)(x^2y-13)$.
49. $(p^2q^2r^2+13)(p^2q^2r^2+16)$.
50. $(a^4b^4c^4+0.7)(a^4b^4c^4+0.3)$.
51. $(p^2q+5.2r)(p^2q+4.8r)$.
52. $(4ax+7)(4ax+39)$.
53. $(5abc+16)(5abc+14)$.
54. $(23m^2+11)(23m^2+12)$.
55. $(41a^4+17)(17-41a^4)$.
56. $(31m^3+1)(1-31m^3)$.

25. Products of Trinomials. Since we may express the trinomial $a+b+c$ in a binomial form, as $(a+b)+c$ or $a+(b+c)$, we may often apply the third principle of § 24 in finding the product of two trinomials.

For example,

$$\begin{aligned}(a+b+c)(a-b-c) &= [a+(b+c)][a-(b+c)]\\ &= a^2-(b+c)^2\\ &= a^2-b^2-2bc-c^2.\\ (x+y-z)(x-y+z) &= [x+(y-z)][x-(y-z)]\\ &= x^2-(y-z)^2\\ &= x^2-y^2+2yz-z^2.\end{aligned}$$

Exercise 17. Products of Certain Trinomials

Examples 1 to 9, oral — Examples 10 to 16, written

1. State two ways in which $x+y-z$ may be written in the form of a binomial. Do the same for $x-y+z$.

2. How could $(m+n+3)(m+n-3)$ be considered as the product of two binomials? Find this product.

3. Multiply $a+b+5$ by $a+b-5$.

Multiply as indicated:

4. $(m+n+2)(m+n-2)$.

5. $(p+q+7)(p+q-7)$.

6. $(x+y+z)(x+y-z)$.

7. $(a+b+9)(a+b-9)$.

8. $(a+b^2+1)(a+b^2-1)$.

9. $(a+b^2+3)(a+b^2-3)$.

10. Write $(1+x-y)(1-x+y)$ so that it shall appear as the product of two binomials, and find this product.

Multiply as indicated:

11. $(9+x+y)(9+x-y)$.

12. $(a-7+b)(a+7-b)$.

13. $(x+2y-z)(x-2y+z)$.

14. $(x^2-xy+y^2)(x^2+xy+y^2)$.

15. $(p^2+q^2-pq)(p^2+q^2+pq)$.

16. $(a^2-2ax+4x^2)(a^2+2ax-4x^2)$.

26. Special Quotients. The following special quotients have been studied in elementary algebra:

1. $\dfrac{a^2+2ab+b^2}{a+b}=a+b.$ 2. $\dfrac{a^2-2ab+b^2}{a-b}=a-b.$

3. $\dfrac{a^2-b^2}{a+b}=a-b,$ and $\dfrac{a^2-b^2}{a-b}=a+b.$

4. $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$ 5. $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2$

Exercise 18. Special Quotients

1. Write out the statements of the above five divisions like the five laws on page 31.

Write out the quotients of the following:

2. $\dfrac{0.04m^2-1}{0.2m+1}.$

3. $\dfrac{49a^2b^2-1}{7ab-1}.$

4. $\dfrac{121m^2-25}{11m+5}.$

5. $\dfrac{49x^2y^2-14xy+1}{7xy-1}.$

6. $\dfrac{x^2+4xy+4y^2-1}{x+2y-1}.$

7. $\dfrac{9a^2+12ab+4b^2-25}{3a+2b+5}.$

8. $\dfrac{16p^2-24pq+9q^2-36}{4p-3q-6}.$

9. $\dfrac{36x^2+48xy+16y^2-49}{6x+4y+7}.$

10. $\dfrac{49a^2+56ab+16b^2-64}{7a+4b-8}.$

11. $\dfrac{81x^2-90xy+25y^2-121}{9x-5y+11}.$

12. $\dfrac{a^3+3a^2b+3ab^2+b^3+x^3+3x^2y+3xy^2+y^3}{a+b+x+y}.$

13. $\dfrac{a^3-3a^2b+3ab^2-b^3-x^3+3x^2y-3xy^2+y^3}{a-b-x+y}.$

14. Divide $(x+y)^2+2(x+y)+1$ by $x+y+1.$

15. Divide $(x-y)^2+2(x-y)+1$ by $x-y+1.$

16. Divide $(x+y)^2-(x-y)^2$ by $(x+y)-(x-y)$, without simplifying either expression. Then simplify each and divide.

CHAPTER II

FACTORS

27. Factor. Any one of two or more numbers which multiplied together form a product is called a *factor* of the product.

For example, 2 and 3 are factors of 6; 2, π, and r are factors of $2\pi r$, and $(b + b')$ and h are factors of $(b + b')h$.

A factor containing a letter is called a *literal factor;* one that is expressed by a numeral is called a *numerical factor.* In the expression $2a(4 + c)$, 2 is a numerical factor; a and $4 + c$ are literal factors.

Except where the contrary is stated, factors are limited to expressions that do not contain fractions or indicated roots. For example, although $3 \cdot \frac{2}{3} = 2$, and $n \cdot \frac{a}{n} = a$, we do not speak of 3 and $\frac{2}{3}$ as factors of 2, nor of n and $\frac{a}{n}$ as factors of a. In the same way, although $(\sqrt{x} + \sqrt{a})(\sqrt{x} - \sqrt{a}) = x - a$ (§ 24), we would not, at this stage, speak of $x - a$ as factorable. If fractions occur as coefficients, they are admitted as factors. For example, $\frac{2}{3}a + \frac{2}{3}b = \frac{2}{3}(a + b)$.

28. Prime and Factorable Expressions. An expression that contains no factors except itself and unity is said to be *prime.*

An expression that contains factors other than itself and unity is said to be *factorable.*

In factoring an expression we separate it into its prime factors. Thus the factors of a^3 are given as a, a, and a, although a^2 is also a factor of a^3.

29. Reduction of Fractions. We use factoring in reducing fractions to lowest terms, exactly as in arithmetic.

Just as $\frac{6}{15}$ is reduced to $\frac{2}{5}$ by canceling the common factor 3, that is, by dividing both terms by 3, so $\frac{a^2}{ab}$ is reduced to $\frac{a}{b}$ by canceling the common factor a. The subject is more fully treated in Chapter III.

30. Monomial Factors of Polynomials. When the terms of a polynomial have a common monomial factor the expression may be factored by inspection. Thus,

$$ax + bx - cx = (a + b - c)x.$$

31. Polynomials factored by grouping Terms. If we do not find a monomial factor that is common to all terms of a polynomial, we may frequently factor by grouping the terms. Thus,

$$ax + ay + bx + by = a(x + y) + b(x + y)$$
$$= (a + b)(x + y).$$

Therefore $ax + ay + bx + by = (a + b)(x + y)$.

Exercise 19. Monomial and Binomial Factors

Factor the following:

1. $a^4 + a^3b - a^2b^2$.
2. $x^5 - 3x^4 + 7x^3$.
3. $m^4 + 3m^3 + 2m^2$.
4. $9p^2q + 6pq^2 - 3pq$.
5. $8a^3b - 6ab^3 + 4a^2b^2$.
6. $5a^5b^2 + 5a^2b^5 - 10a^2b^2$.
7. $4a^5b^4 - 6a^4b^3 + 8a^3b^2$.
8. $17x^2y^3z^4 + 187x^4y^3z^2$.
9. $-19a^2b^2x^5y^5 - 95abx^6y^6$.
10. $76m^3n^3x^2y^2 - 228m^5n^2x^2y^6$.
11. $a^3 + a^2b + a + b$.
12. $a^2 + ab + a + b$.
13. $m^2 - am + m - a$.
14. $p^2q^2 + pq + pqr + r$.
15. $a^2 + ab + ac + bc$.
16. $x^3 + xy + x^2y^2 + y^3$.
17. $a^3b^3 + ab + a^2b^2 + 1$.
18. $p^3 + p^2q + pq^2 + q^3$.
19. $p^5 + pq^4 - p^4q - q^5$.
20. $6xy + 9x + 4y + 6$.
21. $m^2x + m^2y + m^2z + x + y + z$.
22. $a^2bm + a^2bn + a^2bp + 2m + 2n + 2p$.

After factoring the numerator reduce to lowest terms:

23. $\dfrac{a^2 + 2ab + a + 2b}{a + 1}$.
24. $\dfrac{ab^2 + b^2 + 2a + 2}{b^2 + 2}$.
25. $\dfrac{x^3y^3 + x^2y^2 + xy + 1}{xy + 1}$.
26. $\dfrac{x^2y^2z^2 - x^3y^3z^3 + 1 - xyz}{1 - xyz}$.

32. Trinomials that are Perfect Squares. We have learned (§ 24) that a trinomial is a perfect square if it is of the form $a^2 + 2ab + b^2$ or the form $a^2 - 2ab + b^2$.

That is, $\quad a^2 + 2ab + b^2 = (a + b)^2$,

and $\quad a^2 - 2ab + b^2 = (a - b)^2$.

Therefore *a trinomial, arranged according to the descending powers of one of its letters, is a perfect square if the first and last terms are positive and are perfect squares and the middle term is plus or minus twice the product of the square roots of the first and last terms.*

To factor a trinomial that is a perfect square, arrange it according to the powers of some letter and extract the square root of the first and last terms, and connect these square roots by the sign of the middle term. The result is one of the two equal factors.

Exercise 20. Perfect Squares

In the following replace the parentheses by a term that will make the trinomial a perfect square:

1. $x^2 + 2x + (\quad)$.
2. $(\quad) + 2mn + n^2$.
3. $p^2 + (\quad) + q^2$.
4. $a^2 - 2ax + (\quad)$.
5. $25x^2 - (\quad) + 4$.
6. $(\quad) - 8xy + y^2$.
7. $9x^2 - 12xy + (\quad)$.
8. $9x^2 - (\quad) + 16y^2$.

Write the factors of the following:

9. $49a^2 + 14ab + b^2$.
10. $49a^2 - 14ab + b^2$.
11. $81a^2 + 36ab + 4b^2$.
12. $81a^2 - 54ab + 9b^2$.
13. $121x^2 + 22xy + y^2$.
14. $121x^2 - 44xy + 4y^2$.
15. $121x^2 + 66xy + 9y^2$.
16. $121x^2 - 88xy + 16y^2$.
17. $49x^4y^4 + 56x^2y^2 + 16$.
18. $49x^4y^4 - 28x^2y^2 + 4$.
19. $49x^2y^2 + 42xyz + 9z^2$.
20. $64x^2y^2 - 80xyz + 25z^2$.
21. $169p^4 - 182p^2q^2 + 49q^4$.
22. $225m^6 + 30m^3 + 1$.

33. Difference of Two Squares. Since $(a+b)(a-b)=a^2-b^2$ (§ 24), we have

$$a^2-b^2=(a+b)(a-b).$$

The factors of the difference of the squares of two quantities are the sum of the quantities and their difference.

1. Factor $36m^2-25$.

$$36m^2-25=(6m+5)(6m-5).$$

2. Factor a^4-b^4.

$$a^4-b^4=(a^2+b^2)(a^2-b^2).$$
$$=(a^2+b^2)(a+b)(a-b).$$

3. Factor x^8-y^8.

$$x^8-y^8=(x^4+y^4)(x^4-y^4)$$
$$=(x^4+y^4)(x^2+y^2)(x+y)(x-y).$$

Exercise 21. Factoring the Difference of Two Squares

Factor the following:

1. x^4-y^4.	9. $0.0016x^4-1$.	17. x^8-1.
2. x^4-1.	10. $1-0.0016x^4$.	18. x^8-256.
3. $1-x^4$.	11. $0.0016x^4-y^4$.	19. x^8y^8-1.
4. x^4y^4-1.	12. $0.0081x^4-1$.	20. $x^8y^8-z^8$.
5. $1-x^4y^4$.	13. $81x^4-0.0016$.	21. $x^8y^8z^8-1$.
6. x^4-16.	14. $81x^4-16y^4$.	22. $1-x^8y^8z^8$.
7. $16-x^4$.	15. $16x^4-0.0081$.	23. $x^8y^8-a^8b^8$.
8. x^4-16y^4.	16. $16x^4-0.0081y^4$.	24. $x^8y^8z^8-a^8$.

Factor the numerator and denominator. Then reduce the fraction to lowest terms by canceling common factors from numerator and denominator.

25. $\dfrac{a^2-1}{a^4-1}$.

26. $\dfrac{a+1}{a^4-1}$.

27. $\dfrac{a-1}{a^4-1}$.

28. $\dfrac{1-x^4}{1+x^2}$.

29. $\dfrac{x+y}{x^4-y^4}$.

30. $\dfrac{x-y}{x^4-y^4}$.

34. Difference of Two Squares. Special Case. One of the terms in § 33 may be the square of a polynomial.

1. Factor $(a + 2b)^2 - 9c^2$.

Taking the square roots of $(a + 2b)^2$ and $9c^2$, and proceeding as on page 38, we have

$$(a + 2b)^2 - 9c^2 = (a + 2b + 3c)(a + 2b - 3c).$$

2. Factor $x^4 - (a - b)^2$.

$$x^4 - (a - b)^2 = [x^2 + (a - b)][x^2 - (a - b)]$$
$$= (x^2 + a - b)(x^2 - a + b).$$

Similarly, both of the terms may be squares of polynomials.

Thus, $(a + b)^2 - (c - d)^2 = [(a + b) + (c - d)][(a + b) - (c - d)]$
$$= (a + b + c - d)(a + b - c + d).$$

Exercise 22. Factoring the Difference of Two Squares

Examples 1 to 7, oral — Examples 8 to 19, written

1. Factor $a^2 - x^2$; $(a + b)^2 - x^2$; $(a - b)^2 - x^2$.
2. Factor $(a+b+c)^2-x^2$; $(a+b-c)^2-x^2$; $(a-b+c)^2-x^2$.
3. Factor $(2a + b)^2 - c^2$; $(2a - b)^2 - c^2$; $(a + 2b)^2 - c^2$.

Factor the following:

4. $(x + y)^2 - z^2$.
5. $(x - y)^2 - z^2$.
6. $(m + n)^2 - x^2$.
7. $(m - n)^2 - x^2$.
8. $a^2 - (b + c)^2$.
9. $a^2 - (b - c)^2$.
10. $4a^2 - (b + c)^2$.
11. $(a - b)^2 - (c + d)^2$.
12. $(a - b)^2 - (c - d)^2$.
13. $(a + 3b)^2 - (c + 3d)^2$.
14. $(3x - y)^2 - (3a - b)^2$.
15. $(a + b)^2 - (c + d + e)^2$.
16. $(x-y)^2-(2m+3n-p)^2$.
17. $(x - 7)^2 - (a + b + c)^2$.

Reduce to lowest terms as on page 38:

18. $\dfrac{a^2 - (b + c)^2}{(a + b + c)^2}$.
19. $\dfrac{(a + b)^2 - (c + d)^2}{a + b + c + d}$.

35. Difference of Two Squares. Special Case. By properly grouping the terms, a polynomial may often be factored as the difference of two squares.

1. Factor $x^2 + 2xy + y^2 - 16z^2$.

$$x^2 + 2xy + y^2 - 16z^2 = (x + y)^2 - 16z^2$$
$$= (x + y + 4z)(x + y - 4z).$$

2. Factor $9x^2 - 9z^2 - y^2 - 6yz$.

We see that the first term is the square of $3x$, and that the last three terms contain y and z. Grouping and changing the signs of the terms placed in parentheses preceded by $-$, we have

$$9x^2 - (9z^2 + 6yz + y^2) = 9x^2 - (3z + y)^2$$
$$= [3x + (3z + y)][3x - (3z + y)]$$
$$= (3x + 3z + y)(3x - 3z - y).$$

3. Factor $a^2 + b^2 - x^2 - y^2 - 2ab + 2xy$.

Grouping the terms containing a and b, and then those containing x and y, we have

$$(a^2 - 2ab + b^2) - (x^2 - 2xy + y^2) = (a - b)^2 - (x - y)^2$$
$$= [(a - b) + (x - y)][(a - b) - (x - y)]$$
$$= (a - b + x - y)(a - b - x + y).$$

Exercise 23. Factoring the Difference of Two Squares

Examples 1 to 3, oral — Examples 4 to 11, written

1. Find the square of a binomial by selecting a group of three terms from $a^2 - x^2 + b^2 - 2ab$.

2. Find the same (see Ex. 1) in $a^2 + 4b^2 - y^2 + 4ab$.

3. Find the same (see Ex. 1) in $4a^2 - x^2 + 9b^2 + 12ab$.

Factor the following:

4. $a^2 - 2ab - c^2 + b^2$.

5. $a^2 - c^2 + b^2 + 2ab$.

6. $4x^2 - 16y^2 + 1 + 4x$.

7. $1 - x^2 + 2xy - y^2$.

8. $x^2 - a^2 - b^2 - 2ab$.

9. $a^2 - 6ab - 4c^2 + 9b^2$.

10. $4x^2 - 12xz + 12yw + 9z^2 - 4w^2 - 9y^2$.

11. $p^4 - q^2 - 6p^2 - 4m^4 + 9 + 4qm^2$.

36. Difference of Two Squares. Special Case. A case of factoring, occasionally required in courses of study, and merely a special case of the difference of two squares, is illustrated in the following examples:

1. Factor $x^4 + x^2y^2 + y^4$.

If the middle term were $2x^2y^2$ we should have the square of $x^2 + y^2$. Therefore if we add x^2y^2 and also subtract it so as not to change the value, we shall have the difference of two squares, thus:

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy). \\ \therefore\ x^4 + x^2y^2 + y^4 &= (x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

2. Factor $x^4 + 4x^2y^2 + 16y^4$.

The middle term should be $8x^2y^2$ to make the trinomial a perfect square. Therefore

$$\begin{aligned} x^4 + 4x^2y^2 + 16y^4 &= x^4 + 8x^2y^2 + 16y^4 - 4x^2y^2 \\ &= (x^2 + 4y^2)^2 - 4x^2y^2 \\ &= (x^2 + 4y^2 + 2xy)(x^2 + 4y^2 - 2xy). \\ \therefore\ x^4 + 4x^2y^2 + 16y^4 &= (x^2 + 2xy + 4y^2)(x^2 - 2xy + 4y^2). \end{aligned}$$

Exercise 24. Factoring

Examples 1 and 2, oral — Examples 3 to 12, written

1. What must be added to the middle term of $x^4 + 3x^2y^2 + 4y^4$ to make the trinomial a perfect square?

2. What must be added to the trinomials $x^4 - 5x^2y^2 + 4y^4$ and $x^4 + 4y^4$ to make them perfect squares?

Factor the following:

3. $p^4 + p^2 + 1$.

4. $x^4 - 7x^2 + 1$.

5. $16a^4 + 4a^2 + 1$.

6. $9a^4 + 26a^2b^2 + 25b^4$.

7. $9m^4 - 15m^2 + 1$.

8. $x^4 + 4$.

9. $x^4 - 7x^2y^2 + y^4$.

10. $9m^4 + 11m^2 + 4$.

11. $4a^4 - 29a^2b^2 + 25b^4$.

12. $625a^4 + 100a^2b^2 + 16b^4$.

37. The Quadratic Trinomial. A trinomial of the form $ax^2 + bx + c$ is called a *quadratic trinomial.* We shall first consider the special case in which $a = 1$, namely,

$$x^2 + bx + c.$$

We first study the product of two binomials.

$$\begin{array}{r} x + 5 \\ x + 9 \\ \hline x^2 + 5x \\ \quad 9x + 45 \\ \hline x^2 + 14x + 45 \end{array} \qquad \begin{array}{r} x + p \\ x + q \\ \hline x^2 + px \\ \quad qx + pq \\ \hline x^2 + (p+q)x + pq \end{array}$$

By examining the coefficients and the absolute term we derive the following rule for finding the factors of an expression in the form of $x^2 + bx + c$:

Find two monomials whose algebraic product is the absolute term with its proper sign, and whose algebraic sum is the coefficient of x with its proper sign.

Write for the factors two binomials, the first term of each being x, and the second terms being, respectively, the monomials thus found.

We should notice that when c is negative the two monomials sought have unlike signs.

When c is positive the monomials have each the same sign as the middle term, bx.

1. Factor $x^2 - 5x - 14$.

The absolute term, which is the product of the two numbers sought, being negative, the two numbers have unlike signs.

The algebraic sum being negative, the negative number has the greater numerical value.

The two numbers whose product is -14 and whose sum is -5 are -7 and $+2$.

Therefore $x^2 - 5x - 14 = (x - 7)(x + 2)$.

2. Factor $x^2 - 19x + 48$.

Since $-3 + (-16) = -19$, and $-3(-16) = 48$, the factors are evidently $(x - 3)(x - 16)$.

Exercise 25. The Quadratic Trinomial

Factor the following:

1. $x^2 + 23x + 132$.
2. $x^2 - 3x - 88$.
3. $x^2 + 10x + 21$.
4. $x^2 + 20x + 99$.
5. $x^2 + 15x + 54$.
6. $x^2 - 5x - 84$.
7. $m^2 - mn - 156n^2$.
8. $m^2 + 10mn - 39n^2$.
9. $p^2 + 11pq^2 - 60q^4$.
10. $p^2 - 11pq^3 - 60q^6$.
11. $a^2 + 8a + 15$.
12. $a^2 + 10a + 24$.
13. $x^2 - 5x - 14$.
14. $x^2 - 5x - 6$.
15. $p^2 - p - 6$.
16. $p^2 - 13p + 40$.
17. $n^2 - 16n + 63$.
18. $n^2 - 13n + 22$.
19. $m^2 - 2m - 15$.
20. $b^2 + 2b - 24$.
21. $b^2 - 5b - 14$.
22. $t^2 + 5t + 6$.
23. $a^2 + 6a - 55$.
24. $p^2 - 8pq - 65q^2$.
25. $x^2 + 4xy - 21y^2$.
26. $x^2 - xy - 72y^2$.
27. $x^2 + xy - 56y^2$.
28. $x^2 + 25x + 156$.
29. $x^2 + 3x - 130$.
30. $x^2 - 2x - 143$.
31. $x^2 + 5x - 150$.
32. $x^2 - 4x - 165$.
33. $x^2 + 30xy + 200y^2$.
34. $x^2 - 10xy - 200y^2$.
35. $x^2 - xy - 132y^2$.
36. $m^2 + mn - 132n^2$.
37. $p^2 + 2pq - 143q^2$.
38. $p^2 + 3pq^2 - 154q^4$.
39. $a^2 - 7ab^2 - 260b^4$.
40. $x^4 + 7x^2y - 260y^2$.
41. $a^4 - 15a^2b - 450b^2$.
42. $a^6 - 25a^3b - 600b^2$.
43. $x^2 - 34xy - 800y^2$.
44. $x^2 - 29xy - 1050y^2$.

Factor both terms and then reduce to lowest terms:

45. $\dfrac{x^2 + 8x + 15}{x^2 - 2x - 35}$.
46. $\dfrac{a^2 + 10a + 24}{a^2 - a - 42}$.
47. $\dfrac{p^2 + 20p + 99}{p^2 - 3p - 108}$.
48. $\dfrac{x^2 + 22x + 121}{x^2 - x - 132}$.

38. The General Quadratic Trinomial. Having considered the special case of $x^2 + bx + c$ we now consider the general case, namely,

$$ax^2 + bx + c.$$

In elementary algebra we commonly find the factors by trial, as shown in the following examples:

1. Factor $15x^2 - 11x - 14$.

The factors of $15x^2$ are $5x$ and $3x$, or $15x$ and x.

The factors of 14 are 7 and 2, or 14 and 1; and since we have -14, one of these must be negative.

By placing the numbers as shown at the right we can easily see that $5 \cdot 2 - 3 \cdot 7 = -11$, and hence that the factors of the trinomial are $5x - 7$ and $3x + 2$.

5	-7
3	2

Check. Letting $x = 1$ we have $5x - 7 = -2$, $3x + 2 = 5$, and $-2 \cdot 5 = -10$. Also, $15x^2 - 11x - 14 = -10$.

2. Factor $77x^2 + 41xy - 10y^2$.

We evidently have $(7x \pm 5y)(11x \mp 2y)$, where the upper signs go together and the lower also go together; or we may have $(7x \pm 10y)(11x \mp y)$, and so on.

7	5
11	-2

Arranging the numbers as in Ex. 1, we easily see that $(7x + 5y)(11x - 2y)$ give the required product.

We check the work as in Ex. 1.

3. Factor $6x^2 - 77x + 221$.

The factors of $6x^2$ are $3x$ and $2x$, or $6x$ and x.

The factors of 221 are 13 and 17, or 221 and 1, and both must be negative to have $-77x$ and $+221$.

Evidently -1 and -221 cannot be used, for they would make the second term too large.

Of $(3x - 13)(2x - 17)$ or $(3x - 17)(2x - 13)$, it is easily seen the first gives the required product.

4. Factor $32x^2 + 867x + 81$.

The factors of $32x^2$ are $32x$ and x, $16x$ and $2x$, or $8x$ and $4x$.

The factors of 81 are 81 and 1, 27 and 3, or 9 and 9.

The middle term tells us that we must choose large factors, but evidently not 32 with 81, or 16 with 27, because, as here shown, the sum of the cross products would not be large enough.

32	81
1	1

By a little thought we see that the factors are $32x + 3$ and $x + 27$.

39. The General Quadratic Trinomial continued. There are several other methods of factoring a trinomial of the type $ax^2 + bx + c$. The teacher may substitute one of the following for the trial method already given:

1. *By splitting the middle term.* Since

$$(mx + n)(px + q) = mpx^2 + (np + mq)x + qn,$$

we see that the coefficient of x is the sum of two numbers (np and mq) whose product is the product of the coefficient of x^2 (that is, mp) and the absolute term (that is, qn).

Consider the trinomial $10x^2 - x - 21$.

What are the two numbers whose sum is -1 and whose product is $10(-21)$, or -210? Evidently -15 and 14.

Then
$$\begin{aligned} 10x^2 - x - 21 &= 10x^2 + 14x - 15x - 21 \\ &= 2x(5x + 7) - 3(5x + 7) \\ &= (2x - 3)(5x + 7). \end{aligned}$$

That is, to factor the type $ax^2 + bx + c$,

Find two numbers whose algebraic sum is b and product ac. Separate b into these parts and factor by grouping.

2. *By making the first term a square.* We proceed as follows:

$$\begin{aligned} 10x^2 - x - 21 &= \frac{100x^2 - 10x - 210}{10} \\ &= \frac{(10x)^2 - (10x) - 210}{10} \\ &= \frac{(10x - 15)(10x + 14)}{10} \\ &= \frac{5(2x - 3) \cdot 2(5x + 7)}{10} \\ &= (2x - 3)(5x + 7). \end{aligned}$$

3. *By substitution.* Taking the above example,

let
$$y = 10x, \text{ or } \tfrac{1}{10}y = x.$$

Then
$$\begin{aligned} 10x^2 - x - 21 &= \tfrac{1}{10}y^2 - \tfrac{1}{10}y - 21 \\ &= \tfrac{1}{10}(y - 15)(y + 14) \\ &= \tfrac{1}{10}(10x - 15)(10x + 14) \\ &= (2x - 3)(5x + 7). \end{aligned}$$

4. *By formula.* It is possible to factor an expression of the type $ax^2 + bx + c$ by formula, without any trial.

$$ax^2 + bx + c = \frac{a^2x^2 + bax + ac}{a}$$

$$= \frac{(ax)^2 + b(ax) + \frac{b^2}{4} + ac - \frac{b^2}{4}}{a}$$

$$= \frac{\left(ax + \frac{b}{2}\right)^2 - \left(\frac{b^2}{4} - ac\right)}{a}$$

$$= \frac{\left(ax + \frac{b}{2} + \sqrt{\frac{b^2}{4} - ac}\right)\left(ax + \frac{b}{2} - \sqrt{\frac{b^2}{4} - ac}\right)}{a}.$$

For example, consider $10x^2 - x - 21$.

Here $a = 10$, $b = -1$, $c = -21$. Hence the factors are

$$\frac{(10x - \frac{1}{2} + \sqrt{\frac{1}{4} + 210})(10x - \frac{1}{2} - \sqrt{\frac{1}{4} + 210})}{10}$$

$$= \frac{(10x - \frac{1}{2} + \sqrt{\frac{841}{4}})(10x - \frac{1}{2} - \sqrt{\frac{841}{4}})}{10}$$

$$= \frac{(10x - \frac{1}{2} + \frac{29}{2})(10x - \frac{1}{2} - \frac{29}{2})}{10}$$

$$= \frac{(10x + 14)(10x - 15)}{10} = (5x + 7)(2x - 3).$$

5. *By quadratics.* Having studied quadratic equations, the class may proceed as follows in the factoring of $10x^2 - x - 21$:

Consider what values of x will make $10x^2 - x - 21 = 0$.

If $10x^2 - x - 21 = 0$,

then $x^2 - \frac{1}{10}x + \frac{1}{400} = \frac{21}{10} + \frac{1}{400} = \frac{841}{400}$.

$\therefore x - \frac{1}{20} = \pm \frac{29}{20}$,

and $x = \frac{1}{20} \pm \frac{29}{20} = \frac{3}{2}$ or $-\frac{7}{5}$.

Hence $x - \frac{3}{2} = 0$, or $x + \frac{7}{5} = 0$,

and $(x - \frac{3}{2})(x + \frac{7}{5}) = 0$,

or $(2x - 3)(5x + 7) = 0 = 10x^2 - x - 21$.

An advanced class may properly study all five of these methods, in addition to the one given on page 44.

Exercise 26. Factoring the General Quadratic Trinomial

Factor the following:

1. $6a^2 + 17a + 12$.
2. $14a^2 - 11a - 15$.
3. $12m^2 - 5m - 2$.
4. $10x^2 + x - 21$.
5. $10x^2 - 29x - 21$.
6. $10x^2 + 29x - 21$.
7. $6x^2 - 37x + 55$.
8. $16m^2 - 56mn + 49n^2$.
9. $6x^2 - 5x - 6$.
10. $8x^2 - 2x - 15$.
11. $2x^2 + 5x - 3$.
12. $3x^2 - 7x - 6$.
13. $15a^2 + ab - 2b^2$.
14. $9x^2 + 3xy - 2y^2$.
15. $8a^2 - 50a - 93$.
16. $3x^2 + 4xy - 7y^2$.
17. $12a^2 + 49a + 44$.
18. $12a^2 + 38a - 22$.
19. $4m^2 + 71m - 18$.
20. $35a^2 + 86a + 48$.
21. $35a^2 + 26a - 48$.
22. $35a^2 - 86ab + 48b^2$.
23. $22m^2 + 13mn + n^2$.
24. $22m^2 - 9mn - n^2$.
25. $16x^2 + 17x + 1$.
26. $16x^2 + 18x + 2$.
27. $16x^2 - 21x + 5$.
28. $16x^2 - 31x + 15$.
29. $12a^2 - 37a + 21$.
30. $12a^2 + 37a + 21$.
31. $12a^2 - a - 1$.
32. $6m^2 + 13m + 5$.
33. $8a^2 - 22a - 21$.
34. $9p^2 + 30pq + 25q^2$.
35. $8a^2 + 14ab - 15b^2$.
36. $6x^2 + 19xy - 7y^2$.
37. $11a^2 - 23ab + 2b^2$.
38. $21x^2 + 26x - 15$.
39. $70x^2 - 27x - 9$.
40. $3x^2 + 2xy - y^2$.
41. $4x^2 - 12ax + 9a^2$.
42. $6x^2 - 17xy - 14y^2$.
43. $6x^2 + 13xy + 6y^2$.
44. $3x^2 - 11xy + 6y^2$.
45. $12m^2 - 50m + 22$.
46. $4p^2q^2 + 85pq + 81$.
47. $21a^2 + 22a + 1$.
48. $24a^2 - 70ab - 75b^2$.
49. $12x^3 - 37x^2 - 44x$.
50. $9x^2y^4 - 30xy^2z + 25z^2$.
51. $6b^2x^2 - 7bx^3 - 3x^4$.
52. $4a^2x^2 - 8abx + 3b^2$.
53. $6a^2b^2 - ab^3 - 12b^4$.
54. $9x^2y^2 - 3xy^3 - 6y^4$.
55. $2x^4 - 4x^3y - 6x^2y^2$.
56. $3a^2 - 21ab + 30b^2$.

40. Sum or Difference of Two Cubes. By multiplication of the two factors we see that

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

and

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Therefore, *the factors of the sum of the cubes of two quantities are* (1) *the sum of the quantities;* (2) *the square of the first, minus the product of the first and second, plus the square of the second.*

The factors of the difference of the cubes of two quantities are (1) *the difference of the quantities;* (2) *the square of the first, plus the product of the first and second, plus the square of the second.*

Factor $27x^3 + y^3$.

$$27x^3 + y^3 = (3x)^3 + y^3 = (3x + y)(9x^2 - 3xy + y^2).$$

41. Factoring Perfect Cubes. We have found (§ 24) that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Therefore *the cube root of a polynomial in the form* $a^3 \pm 3a^2b + 3ab^2 \pm b^3$ *is of the form* $a \pm b$.

When there is the double sign, $\pm$, in both members, the upper signs go together and the lower signs go together.

For example, $8x^3 + 12x^2 + 6x + 1 = (2x + 1)^3$.

Exercise 27. Factors involving Cubes

Factor the following:

1. $x^3y^3 + 512$.
2. $8m^3 - 512(x + y)^3$.
3. $8(a + b)^3 + 512(x + y)^3$.
4. $729 + 27(a + b)^3$.
5. $(x + 1)^3 + (y + 1)^3$.
6. $(a + y)^3 - (x + b)^3$.
7. $p^3 - 6p^2 + 12p - 8$.
8. $8x^3 + 36x^2 + 54x + 27$.
9. $8x^3 - 12x^2 + 6x - 1$.
10. $64p^3 + 48p^2 + 12p + 1$.
11. $64p^3 + 12p - 48p^2 - 1$.
12. $a^3 - 12a^2b + 48ab^2 - 64b^3$.

42. The Remainder Theorem. If we divide $x^2 - 7x + 10$ by $x - 2$, the quotient is $x - 5$ and there is no remainder. Furthermore, if we put 2 for x in $x - 2$, we have $2 - 2 = 0$; and if we put 2 for x in the polynomial, we have $2^2 - 7 \cdot 2 + 10 = 0$. That is, the remainder and the polynomial both become zero when we put 2 in place of x.

If we divide $x^2 - 7x + 4$ by $x - 2$, there is a remainder -6, and if we put 2 for x in the polynomial, we have $2^2 - 7 \cdot 2 + 4 = -6$.

In general, if we divide $x^2 + bx + c$ by $x - a$, we see that the remainder is $a^2 + ba + c$, which is the same as the dividend with a substituted for x. That is,

$$\begin{array}{r|l} x^2 + bx + c & x - a \\ \cline{2-2} x^2 - ax & x + a + b \\ \cline{1-1} (a+b)x + c & \\ (a+b)x - a^2 - ba & \\ \cline{1-1} a^2 + ba + c & \end{array}$$

The remainder arising from dividing any integral polynomial in x by $x - a$ is the same expression with a put in place of x.

This is called the Remainder Theorem, the word "theorem" meaning a statement to be proved.

Expressed in the symbols of functions, the remainder arising from dividing $f(x)$ by $x - a$ is $f(a)$.

This important theorem enables us to factor many expressions that do not come under the cases already considered, or to factor the latter more easily.

1. Is $x - 1$ a factor of $x^3 - 7x + 6$?

Substitute 1 for x and we have $1 - 7 + 6 = 0$. Therefore there is no remainder when we divide by $x - 1$. Therefore $x - 1$ is a factor.

2. Is $x - 3$ a factor of $x^2 - 9x + 6$?

Substitute 3 for x and we have $9 - 27 + 6 = -12$. Therefore if we divide by $x - 3$ there is a remainder -12. Therefore $x - 3$ is not a factor. Similarly, $x \pm 1$, $x \pm 2$, and $x + 3$ are not factors.

3. Is $x + 2$ a factor of $x^3 + 4x^2 + 3x - 2$?

Since $x + 2 = x - (-2)$, which is now in the form $x - a$, substitute -2 for x and we have $(-2)^3 + 4(-2)^2 + 3(-2) - 2 = -8 + 16 - 6 - 2 = 0$. Therefore $x + 2$ is a factor.

4. Find the factors of $x^3 + 4x^2 - 11x - 30$.

Since the factors of -30 are $\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15$, we try binomial factors of which these are the second terms.

Trying $x - 1$, $1 + 4 - 11 - 30$ is not zero.
Trying $x + 1$, $-1 + 4 + 11 - 30$ is not zero.
Trying $x - 2$, $8 + 16 - 22 - 30$ is not zero.
Trying $x + 2$, $-8 + 16 + 22 - 30$ is 0.

Therefore $x + 2$ is one factor. Dividing by $x + 2$ we have $x^2 + 2x - 15$, of which we see the factors are $x + 5$ and $x - 3$.

Therefore the factors are $x + 2$, $x - 3$, $x + 5$.

Exercise 28. The Remainder Theorem

Examples 1 to 6, oral — Examples 7 to 21, written

1. Is $x - 1$ a factor of $x^3 - x^2 + x - 1$? of $x^3 - 2x^2 + x$?

2. Is $x - 1$ a factor of $x^2 - 1$? of $x^4 - 1$? of $x^{10} - 1$? of $x^3 - 4x^2 + 4x - 1$? of $x^3 + 7x^2 - 9x + 1$?

3. Is $x - 1$ a factor of $x^{57} - 1$? of $x^{100} - 1$? of $x^{25} + x^{18} - 2$?

4. What remainder arises from dividing $x^{25} + 1$ by $x - 1$?

5. What remainder arises from dividing $x^7 + x^5 + 1$ by $x - 1$? $x^{19} - 2x^{17} + 1$ by $x - 1$?

6. Is $x + 1$ a factor of $x^3 + 1$? of $x^5 + 1$? of $x^7 + 1$? of $x^9 + 1$? of $x^{11} + 1$? of $x^3 + 8$? of $x^2 + 1$?

7. Is $x - 2$ a factor of $x^2 - 4$? of $x^4 - 16$? of $x^6 - 64$? of $x^3 - 9x^2 + 17x - 6$? of $x^3 - 8$? of $x^3 + 8$?

Factor the following:

8. $x^3 - 7x - 6$.
9. $a^3 - 8a + 3$.
10. $x^3 + 7x^2 - 8$.
11. $m^3 - 19m + 12$.
12. $x^3 + 2x^2 - 9x - 18$.
13. $x^3 + 2x^2 - 2x - 1$.
14. $x^3 + x^2 - 4x - 4$.
15. $x^3 - 5x^2 - 2x + 24$.
16. $a^3 - 48ab^2 - 7b^3$.
17. $m^3 - 9mn^2 + 8n^3$.
18. $3p^3 + 4pq^2 - 7q^3$.
19. $x^3 - x^2 - 8x + 12$.
20. $a^3 - 9a^2 + 26a - 24$.
21. $x^3 - 10x^2 + 23x - 14$.

43. Factors of $x^n \pm y^n$. The Remainder Theorem enables us to determine the cases in which $x^n \pm y^n$ is divisible by $x \pm y$.

1. Is $x^n + y^n$ divisible by $x - y$?

Substituting y for x in the binomial $x^n + y^n$ we have $y^n + y^n = 2y^n$. Hence $x^n + y^n$ divided by $x - y$ has a remainder $2y^n$. Hence $x^n + y^n$ is not divisible by $x - y$, whatever may be the value of n.

2. Is $x^n - y^n$ divisible by $x - y$?

Substituting y for x in the binomial $x^n - y^n$ we have $y^n - y^n = 0$. Therefore *$x - y$ is always a factor of $x^n - y^n$.*

In general, in factoring expressions like $x^6 - y^6$ it is better first to take two factors of the same degree, thus:

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3) = (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

This is better than to start with

$$x^6 - y^6 = (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5),$$

for the factors of the second of these factors are not readily seen.

3. Is $x^n + y^n$ divisible by $x + y$?

Since $x + y = x - (-y)$, and this is in the form $x - a$, we substitute $-y$ for x in $x^n + y^n$, and $x^n + y^n$ becomes $(-y)^n + y^n$.

If n is odd, $(-y)^n$ is negative and equals $-y^n$. We then have $-y^n + y^n = 0$. That is, *if n is odd, $x + y$ is a factor of $x^n + y^n$.*

If n is even, $(-y)^n$ is positive and equals y^n. We then have $y^n + y^n = 2y^n$. That is, *if n is even, $x + y$ is not a factor of $x^n + y^n$.*

For example, $x + y$ is a factor of $x^3 + y^3$, but not of $x^2 + y^2$.

Similarly, $x + 2$ is a factor of $x^3 + 8$, but not of $x^4 + 16$.

4. Is $x^n - y^n$ divisible by $x + y$?

Substituting $-y$ for x we have $(-y)^n - y^n$. If n is even, this becomes $y^n - y^n = 0$, because $-y$ raised to any even power is positive. But if n is odd, it becomes $-y^n - y^n = -2y^n$, because $-y$ raised to any odd power is negative. That is, *if n is even, $x + y$ is a factor of $x^n - y^n$, but not if n is odd.*

Summarizing these cases we have the following:

$x^n + y^n$ never has a factor $x - y$;

$x^n - y^n$ always has a factor $x - y$;

$x^n + y^n$ has a factor $x + y$ when n is odd;

$x^n - y^n$ has a factor $x + y$ when n is even.

Exercise 29. Factoring Binomials

Factor the following:

1. $x^3 + y^3$.
2. $x^5 + y^5$.
3. $x^7 + y^7$.
4. $a^9 + y^9$.
5. $a^3 + 8b^3$.
6. $a^3 + 27b^3$.
7. $p^4 + 8p$.
8. $x^5y + y^5$.
9. $32x^5y + y^5$.
10. $32m^5n^2 + n^7$.
11. $x^4 - y^4$.
12. $x^3 - y^3$.
13. $x^{10} - y^{10}$.
14. $25x^2 - 36y^2$.
15. $49a^2 - 81b^2$.
16. $16p^4 - a^4b^4$.
17. $81m^4 - n^4$.
18. $16x^4 - 81y^4$.
19. $a^6 - 64b^6$.
20. $729a^6 - 1$.
21. $x^3 - y^3$.
22. $x^5 - y^5$.
23. $x^7 - y^7$.
24. $8p^3 - 1$.
25. $27m^3 - 1$.
26. $1 - 27a^3b^3$.
27. $64p^3 - 27$.
28. $32p^5 - 1$.
29. $243a^5 - 1$.
30. $243 - 32x^5$.
31. Factor $(x^2 + 2y^2)^2 - (2xy)^2$ and $x^4 + 4y^4$.
32. As in Ex. 31, factor $a^4b^4 + 4c^4$.
33. As in Ex. 31, factor $p^4q^4r^4 + 4x^4y^4$.
34. Factor $(a + b)^3 - (x + y)^3$, as the difference of two cubes.

Factor the following:

35. $a^3 + (b + c)^3$.
36. $a^3 - (b + c)^3$.
37. $a^3 - (b - c)^3$.
38. $(p + q)^3 - r^3$.
39. $(p - r)^3 + r^3$.
40. $(p - r)^3 + 8r^3$.
41. $a^2 - (b + c)^2$.
42. $a^4 - (b + c)^4$.
43. $8a^3 - 27b^3c^3d^3e^3$.
44. $m^5 + 32p^5q^5r^5$.
45. $x^7y^7z^7 - a^7b^7c^7d^7$.
46. $128m^7n^7 - a^7b^7x^7y^7$.
47. $(a + b)^2 - (x + y)^2$.
48. $(a - b)^2 - (x - y)^2$.
49. $4(p + q)^2 - 9(a - b)^2$.
50. $25(a - b)^2 - 16(x + y)^2$.
51. $36a^2b^2c^2 - 25(m + n)^2$.
52. $(a + b)^3 + (x + y)^3$.
53. $(a - b)^3 - (x - y)^3$.
54. $8(a + b)^3 - (x + y)^3$.
55. $8(a - b)^3 + (x - y)^3$.
56. $27(p - q)^3 + (m - n)^3$.
57. $(a + b)^5 - 32$.
58. $32(a + b)^5 - 243$.

44. Summary of Factoring. The student will find it of advantage to proceed as follows in factoring an expression:

First remove any monomial factor. Then see if the expression is one of the following forms already studied:

$ax + ay + bx + by$.	(§ 31)
$a^2 \pm 2ab + b^2$.	(§ 32)
$a^2 - b^2$.	(§§ 33–36)
$x^2 + bx + c$.	(§ 37)
$ax^2 + bx + c$.	(§§ 38, 39)
$a^3 \pm b^3$.	(§ 40)
$a^3 \pm 3a^2b + 3ab^2 \pm b^3$.	(§ 41)
$(a - b) \times$ a polynomial.	(§§ 42, 43)

If so, factor as directed under these cases.

Factor each polynomial of the result until every factor is prime.

Students are urged to check their results in factoring, either by the substitution of some value for the letters or by multiplication, and teachers may well afford to insist upon it.

45. Changing the Signs of Factors. Since $abc = a \cdot - b \cdot - c = - a \cdot b \cdot - c = - a \cdot - b \cdot c$, we see that

The signs of any even number of factors may be changed without changing their product.

Therefore $a^2 - 2ab + b^2 = (a - b)(a - b)$ or $(b - a)(b - a)$.

Exercise 30. Review of Factoring

Examples 1 to 5, oral — Examples 6 to 89, written

1. Factor $px + qx$; $mx - my$; $am + am^2$; $ax + bx - cx$.
2. Factor $p^2 + 2p + 1$; $a^2 - 2a + 1$; $1 - 2a + a^2$.
3. Factor $a^2 - 1$; $1 - x^2$; $a^2b^2 - 1$; $a^2b^2 - c^2$.
4. Factor $m^3 + n^3$; $m^3 - n^3$; $m^3 + 1$; $1 + m^3$; $a^3 - 1$; $1 - a^3$.
5. Factor $p^3 + 3p^2 + 3p + 1$; $p^3 - 3p^2 + 3p - 1$.

Factor the following:

6. $a^2 - 16ab^2 + 60b^4$.
7. $27m^3 - 1331$.
8. $27m^3 - 1331x^3y^3$.
9. $a^4 + 5a^3 - 36a^2$.
10. $625x^4y^4 - 81$.
11. $81x^4y^4 - 625$.
12. $a^2 + c - c^2 - a$.
13. $a^3 + a - b - b^3$.
14. $(a + b)^3 - 1$.
15. $1 - (a - b)^3$.
16. $x^{2n} - y^{2n}$.
17. $9x^2 - 24x + 16$.
18. $a^3 + b^2 - a^2 - b^3$.
19. $a^2 + a^4 - b^2 - b^4$.
20. $24m^2 - 35n^2 + 2mn$.
21. $m^2n^2 - 7mn - 44$.
22. $p^2 + 30q^2 + 13pq$.
23. $4a^4 + 8a^2b^2 + 121b^4$.
24. $2ax + 4bx - 10pb - 5pa$.
25. $p^4q^4 - x^4 + p^2q^2 - x^2$.
26. $p^4q^4 - p^2q^2 - x^4 + x^2$.
27. $81a^2 - 180ab + 100b^2$.
28. $x^2y^4 - 12xy^2z + 35z^2$.
29. $64m^2 - 48mn + 9n^2$.
30. $24x^4 - 86x^2y^2 + 42y^4$.
31. $p^2 + 8pq^2 - 33q^4$.
32. $6m^4 + 25m^2n^2 - 91n^4$.
33. $14x^2y^2z^2 + 33 - 83xyz$.
34. $625(a + b)^4 - 81$.
35. $ac - bc - ad + bd$.
36. $xy - xz - y^2 + yz$.
37. $ab - 2a - 2b + 4$.
38. $49x^2 + 56xy + 16y^2 - 16z^2$.
39. $49a^5b - 154a^3b^3 + 121ab^5$.
40. $20ax + 6by - 15ay - 8bx$.
41. $16a^4c^2 + 88a^3c^3 + 121a^2c^4$.
42. $729x^3 + (y + 2z)^3$.
43. $a^4b^2y + a^3b^3x + ab^4z$.
44. $81x^4 - 18x^2y^2 + 49y^4$.
45. $1728x^3 - 1$.
46. $125x^3 - 1728$.
47. $9x^2 - 24xy^4 + 16y^8$.
48. $a^2 - 25b^2 + 9x^2 + 6ax$.
49. $a^3 + a - b^3 - b$.
50. $x^2y^2 + xy - z^2 - z$.
51. $99a^2b^2 - 17abc - 12c^2$.
52. $p^2q^2 - 12pqr + 35r^2$.
53. $x^2y^2z^2 - a^2 + xyz - a$.
54. $x^2y^2z^2 - xyz - a^2 + a$.
55. $x^2y^2z^2 - 15xyz + 56$.
56. $6x^2y^2z^2 - 37xyz + 56$.
57. $6x^2y^2z^2 - xyz - 35$.
58. $6x^2y^2z^2 + 35 - 29xyz$.
59. $35 + 35p^2q^2r^2 - 74pqr$.
60. $86m^2n^2 + 21m^4n^4 + 33$.
61. $14x^2y^2z^2 - 33 - 71xyz$.
62. $27(m + n)^3 - 1331x^3y^3$.
63. $1728(a + b)^3 - 1331$.

Arrange as the difference of two squares, and factor:

64. $a^2 - x^2 - y^2 + 4ab + 2xy + 4b^2$.

65. $a^2 + b^4 - x^4 + 2ab^2 - y^4 + 2x^2y^2$.

66. $4a^2 - 9x^2 + 9b^2 - 16y^2 - 12ab + 24xy$.

67. $49a^2 + 1 - c^4 - 14a - 16b^2 - 8bc^2$.

68. $36(a^2 - c^2) + 12(ab^2 - cd^2) + b^4 - d^4$.

69. $9a^2 - 49d^2 + 4b^2 - 16c^2 + 4(3ab + 14cd)$.

70 $4a^4 + 9b^4 - x^4 - 49 + 12a^2b^2 + 14x^2$.

Arrange as the sum or the difference of two cubes, and factor:

71. $a^3 + 3a^2b + 3ab^2 + b^3 - c^3$.

72. $a^3 - b^3 + c^3 + 3b^2c - 3bc^2$.

73. $8a^3 + 1 - b^3 + 12a^2 + 6a$.

74. $a^3 + b^3 - c^3 - d^3 + 3a^2b - 3c^2d + 3ab^2 - 3cd^2$.

75. $a^3 - c^3 + 8b^3 - 8d^3 + 6a^2b - 6c^2d + 12ab^2 - 12cd^2$.

76. $64a^3 - 48a^2 + 12a - 1 - 125b^3 - 75b^2c - 15bc^2 - c^3$.

Factor the following:

77. $144x^2 - 144ac - 81a^2 - 64c^2$.

78. $x^2 - 4n^2 + 4mn - 6xy - m^2 + 9y^2$.

79. $9x^2 - 9a^2 + 64y^2 - 64b^2 + 48xy + 48ab$.

80. $16x^4 - 81b^2 - 25a^4 + 56x^2y - 90a^2b + 49y^2$.

81. $6a^2 - 9ay - 15by - 20bx - 12ax + 10ab$.

82. $25a^2 - 36x^2 + 121b^2 - 49y^2 - 110ab - 84xy$.

83. $(a + b)^2 - 2(a + b)(c + d) + (c + d)^2$.

84. $(a - b)^2 - 2(a - b)(4c - d) + (4c - d)^2$.

85. $x^3 + 6x^2 + 11x + 6$; also $x^3 - 6x^2 + 11x - 6$.

86. $(a + b + c + d)^2 + 2(a + b + c + d) - 99$.

87. $27(w + x + y + z)^3 - 1331w^3x^3y^3z^3$.

88. $x^3 + y^3 + 3xy(x + y) - (a^3 + b^3) - 3ab(a + b)$.

89. $x^2(x^2 + 4x + 6) + 4x + 1 - 14{,}641$.

46. Application of Factoring to Equations. If we have the equation

$$(x-4)(x+3)=0,$$

one of the factors must be 0, because 0 is the only quantity that can be multiplied by another quantity so as to have the product 0. Hence

either $x-4=0,$

whence $x=4,$

or else $x+3=0,$

whence $x=-3.$

Therefore, to solve an equation by factoring,

Arrange the equation so that one member is zero. Then factor the other member, equate to zero each factor that contains an unknown quantity, and solve the resulting equations.

Solve the equation $x^3-33x+27=-5x^2$.

Rearranging, $x^3+5x^2-33x+27=0.$

Factoring (§ 42), $(x-1)(x-3)(x+9)=0.$

Therefore $x-1=0,$ whence $x=1;$

$x-3=0,$ whence $x=3;$

or $x+9=0,$ whence $x=-9.$

Therefore the roots are 1, 3, and -9.

Exercise 31. Application to Equations

Solve the following equations:

1. $x^2-8x-9=0.$
2. $x^2-8x+7=0.$
3. $x^2-9x+14=0.$
4. $x^2+8x-33=0.$
5. $x^2+5x=36.$
6. $x^2+36=13x.$
7. $x^2+x=56.$
8. $x^2-x=72.$
9. $x^3-7x^2-17x-9=0.$
10. $x^3-6x^2-9x+14=0.$
11. $x^3-7x^2-4x+28=0.$
12. $x^3+9x^2-25x-33=0.$
13. $x^3+4x^2-41x+36=0.$
14. $x^3-14x^2+49x-36=0.$
15. $x^3-57x+56=0.$
16. $x^3-73x-72=0.$

47. Highest Common Factor. The factor of highest degree that is common to two or more expressions is called their *highest common factor* (H.C.F.).

For example, as 4 is the greatest common divisor of 12 and 16, so $a^2 - b^2$ is the highest common factor of $a^2 - b^2$ and $a^4 - b^4$.

48. Finding the Highest Common Factor. The highest common factor may often be found by separating the expressions into factors.

Find the H.C.F. of $ax^4 - ay^4$ and $a^2x^4 - 2a^2x^2y^2 + a^2y^4$.

$$\begin{aligned} ax^4 - ay^4 &= a(x^4 - y^4) \\ &= a(x^2 + y^2)(x^2 - y^2) \\ &= a(x^2 + y^2)(x + y)(x - y). \\ a^2x^4 - 2a^2x^2y^2 + a^2y^4 &= a^2(x^4 - 2x^2y^2 + y^4) \\ &= a^2(x^2 - y^2)^2 \\ &= a^2(x + y)(x - y)(x + y)(x - y). \end{aligned}$$

The factors common to both expressions are a, $x + y$, and $x - y$. Therefore $a(x + y)(x - y)$ is the H.C.F.

Exercise 32. The Highest Common Factor

Find the H.C.F. of the following:

1. $2a^2 + ab - b^2$, $3a^2 + 4ab + b^2$.
2. $3a^2 + 4ab + b^2$, $2a^2 + 5ab + 3b^2$.
3. $5a^2 + 34a - 7$, $6a^2 + 44a + 14$.
4. $6a^2 + 7a + 2$, $8a^2 - 14a - 9$.
5. $a^4 - a^3b$, $ab^4 - b^5$, $b^2 - a^2$.
6. $a^7 + a^6b$, $ab^8 + b^4$, $a^2 + b^2 + 2ab$.
7. $a^2b^2 + ab^3$, $a^4b + a^3b^2$, $a^4(ab + 2b^2) + a^3b^3$.
8. $a(a - 2b) + b^2$, $b^2 - a^2$, $a^3 - b^3$, $a^4 - b^4$.
9. $a^2 + b(b + 2a)$, $a^4 - b^4$, $a^2b^2 + a^3(2b + a)$.
10. $x^4 - 2x^2y^2 + y^4$, $x^2 + y^2 + 2xy$, $(x + y)^3$.
11. $x^3 + 6xy^2 + 5x^2y$, $2x^3y + 5x^2y^2 + 2xy^3$.
12. $2x^4y^2 + 7x^3y^3 - 9x^2y^4$, $2x^3y^3 + 11x^2y^4 + 9xy^5$.
13. $x^2 + 3x + 2$, $x^3 + 6x^2 + 11x + 6$.

49. Highest Common Factor. If two polynomials are not readily factored, their highest common factor may be found by another method. This method depends on the principle that

A factor of each of two quantities is a factor of their sum and their difference.

This is apparent because $ax \pm ay = a(x \pm y)$.

Find the H.C.F. of $2x^3 + x^2 - 12x + 9$ and $2x^3 - 7x^2 + 12x - 9$.

The H.C.F. is a factor of the difference of the two expressions, and therefore of $8x^2 - 24x + 18$.

Since 2 is not a factor of either polynomial, it cannot be a factor of the H.C.F., and hence we divide by 2 and have $4x^2 - 12x + 9$.

Any factor introduced into but one of the polynomials, if it is not already in the other, will not change the H.C.F., since this is composed of *common* factors only. Therefore we multiply the first polynomial by 2 to avoid a fraction in the quotient.

Since the H.C.F. is a factor of $4x^2 - 12x + 9$, it is a factor of x times $4x^2 - 12x + 9$, or of $4x^3 - 12x^2 + 9x$. The H.C.F. is therefore a factor of the difference, $14x^2 - 33x + 18$. The H.C.F. will not be affected if this is multiplied by 2, because 2 is not a factor of either polynomial. Since the H.C.F. is a factor of $4x^2 - 12x + 9$, it is a factor of 7 times $4x^2 - 12x + 9$, or of $28x^2 - 84x + 63$. The H.C.F. is therefore a factor of the difference, $18x - 27$. Since 9 is not a common factor of the polynomials, it may be rejected as a factor of $18x - 27$, and the H.C.F. is therefore a factor of $2x - 3$.

$$
\begin{array}{r|l}
4x^2 - 12x + 9 & 4x^3 + 2x^2 - 24x + 18 \,(x \\
\cline{2-2}
 & \underline{4x^3 - 12x^2 + 9x} \\
 & 14x^2 - 33x + 18 \\
 & \underline{2} \\
 & 28x^2 - 66x + 36 \,(7 \\
 & \underline{28x^2 - 84x + 63} \\
 & 18x - 27 \,)\,9 \\
 & 2x - 3
\end{array}
$$

But $2x - 3$ is prime; therefore if there is any H.C.F., it is $2x - 3$, and by actual division we show that this is a factor of $4x^2 - 12x - 9$.

It is also a factor of $18x - 27$, and therefore of the sum of $18x - 27$ and 7 $(4x^2 - 12x + 9)$, or $28x^2 - 66x + 36$. It is a factor of half of this expression, because it does not contain the factor 2. It is therefore a factor of the sum of $14x^2 - 33x + 18$ and of x $(4x^2 - 12x + 9)$, and of half of this sum, that is, of $2x^3 + x^2 - 12x + 9$.

We therefore see that $2x - 3$ is a factor of the two given expressions, and that it is also the highest common factor.

Practically we rarely have to resort to such a long method of finding the H.C.F. Unless the quantities are much larger than usual we can apply the Remainder Theorem to most cases.

1. Find the H.C.F. of

$$6x^3 + x^2 - 5x - 2 \text{ and } 6x^3 + 5x^2 - 3x - 2.$$

Subtract, and the H.C.F. cannot exceed $4x^2 + 2x$. Suppress the factor $2x$, since this is evidently not a common factor of the polynomials. Then the H.C.F. is $2x + 1$ if there is any H.C.F.

$$\begin{array}{r} 6x^3 + 5x^2 - 3x - 2 \\ 6x^3 + \quad x^2 - 5x - 2 \\ \hline 2x)\overline{4x^2 + 2x} \\ \quad 2x + 1 \end{array}$$

If the expressions are divisible by $2x + 1$, they are divisible by $\frac{1}{2}(2x + 1)$ or by $x + \frac{1}{2}$, or by $x - (-\frac{1}{2})$.

Hence, by the Remainder Theorem, $2x + 1$ is the H.C.F.

2. Find the H.C.F. of $x^4 + x^3 + 2x - 4$ and $x^3 - 2x^2 - 5x + 6$.

By the Remainder Theorem, $x - 1$ is a common factor.

Dividing each by $x - 1$, we have as quotients

$$x^3 + 2x^2 + 2x + 4 = (x + 2)(x^2 + 2),$$

and

$$x^2 - x - 6 = (x + 2)(x - 3).$$

Therefore the H.C.F. is $(x - 1)(x + 2)$.

Exercise 33. Highest Common Factor

Find the highest common factor of:

1. $a^3 + 6a^2 - 8a - 7$ and $a^3 + 8a^2 + 10a + 21$.

2. $4m^4 - 5m^2 + 1$ and $4m^4 + 4m^3 + m^2 - 1$.

3. $p^3 + 5p^2 + 8p + 4$ and $p^3 + 6p^2 + 11p + 6$.

4. $6a^3 + 7a^2b - 22ab^2 - 5b^3$ and $15a^3 - 14a^2b - 13ab^2 - 2b^3$.

5. $a^3 - 5a^2b + 9ab^2 - 9b^3$ and $3a^3 - 5a^2b - 7ab^2 - 15b^3$.

6. $6x^3 - 7x^2 - 16x + 12$ and $4x^3 - 8x^2 - 9x + 18$.

7. $4a^4 + 7a^2 + 16$ and $4a^4 + 12a^3 + 9a^2 - 16$.

8. $2m^3 + m^2n - 4mn^2 - 3n^3$ and $2m^3 + m^2n - 9n^3$.

9. $n^3 + 4n^2 + 5n + 2$ and $n^3 + 2n^2 - n - 2$.

10. $6x^3 - 13x^2 + 19x - 7$ and $9x^3 - 27x^2 + 41x - 28$.

11. $2p^3 - 9p^2 + 11p - 3$ and $4p^3 - 4p^2 - 5p + 3$.

50. Lowest Common Multiple. The multiple of lowest degree that *contains* two or more algebraic expressions as factors is called their *lowest common multiple* (L.C.M.).

For example, 12 is a common multiple of 2 and 3, but 6 is their L.C.M. Similarly, $a^4 - b^4$ is a common multiple of $a + b$ and $a - b$, but $a^2 - b^2$ is their L.C.M.

The L.C.M. is not necessarily the least common multiple for all values of the letters. Thus, if $a = 4$ and $b = 2$, the L.C.M. of $a + b$ and $a - b$, which is $a^2 - b^2$, reduces to $16 - 4 = 12$. The least common multiple of 6 and 2 is, however, 6.

51. Finding the Lowest Common Multiple. From the above definition we see that

The L.C.M. of two or more expressions must contain all the different factors of the expressions, each factor being taken the greatest number of times that it occurs in any of the given expressions.

Find the L.C.M. of $6a^2 + 11ab + 3b^2$ and $4a^2 - 4ab - 15b^2$.

$$6a^2 + 11ab + 3b^2 = (2a + 3b)(3a + b).$$
$$4a^2 - 4ab - 15b^2 = (2a + 3b)(2a - 5b).$$
$$\therefore \text{ the L.C.M.} = (2a + 3b)(3a + b)(2a - 5b).$$

Exercise 34. Lowest Common Multiple

Find the lowest common multiple of:

1. $34a^6b^2c^5$, $51a^5b^5c^5$.
2. $38x^4y^4z^4$, $57x^5y^4z^5$.
3. $58m^2n^4x^9$, $87m^9n^4x^2$.
4. $x^3 + 1$, $x^2 - 1$.
5. $x^3 + x$, $x^4 - 1$.
6. $x^2 + 2x$, $x^2 - 4$.
7. $a^2 + 5ab + 6b^2$, $a^2 + 8ab + 15b^2$.
8. $m^2 + 16mn + 63n^2$, $m^2 - 2mn - 63n^2$.
9. $x^2 - y^2$, $(y - x)^2$, $(x + y)^2$.
10. $x^2 + 9x + 14$, $x^2 + 7x + 10$, $x^2 - 7x - 18$.
11. $y - x$, $x^2 - y^2$, $x^3 - y^3$, $x^4 - y^4$.
12. $a^3 + 3a^2b + 3ab^2 + b^3$, $a^3 + b^3$, $a^2 - b^2$.

52. Finding the Lowest Common Multiple. If the expressions are not readily factored, the L.C.M. may be found by using the H.C.F. as an aid in factoring.

Find the L.C.M. of $2x^4 + 3x^3y - 9x^2y^2$ and $6x^4y - 3xy^4 - 17x^3y^2 + 14x^2y^3$.

By § 49, or by using the Remainder Theorem, the H.C.F. is found to be $x(2x - 3y)$. Using this as one factor, we have

$$2x^4 + 3x^3y - 9x^2y^2 = x^2(2x - 3y)(x + 3y).$$

$$6x^4y - 17x^3y^2 + 14x^2y^3 - 3xy^4 = xy(2x - 3y)(3x^2 - 4xy + y^2)$$
$$= xy(2x - 3y)(3x - y)(x - y).$$

$$\therefore \text{ the L.C.M.} = x^2y(2x - 3y)(x + 3y)(3x - y)(x - y).$$

The L.C.M. of three or more expressions may be obtained by finding the L.C.M. of any two of them; then the L.C.M. of this result and the third expression, and so on.

It is usually as convenient, however, to factor each of the expressions separately and find the L.C.M. in the ordinary manner.

Exercise 35. Lowest Common Multiple

Find the lowest common multiple of:

1. $2x^3 + x^2 - 4x - 3$ and $2x^3 + x^2 - 9$.
2. $a^3 + 4a^2 + 5a + 2$ and $a^3 + 2a^2 - a - 2$.
3. $2x^3 - 9x^2 + 11x - 3$ and $4x^3 - 4x^2 - 5x + 3$.
4. $9a^4 - 4a^2 + 4a - 1$ and $9a^4 - 12a^3 + 4a^2 - 1$.
5. $m^3 - mn^2 + 2m^2n - 2n^3$ and $m^3 - mn^2 - 4m^2n + 4n^3$.
6. $x^3 - x^2y + xy^2 + 14y^3$ and $x^3 - 5x^2y + 13xy^2 - 14y^3$.
7. $9x^2 - 4$, $6x^2 - 13x + 6$ and $6x^2 + 5x - 6$.
8. $x^2 + 2x - 3$, $x^3 + 3x^2 - x - 3$ and $x^3 + 4x^2 + x - 6$.
9. $x^3 + 8x^2 + 19x + 12$ and $x^3 + 9x^2 + 26x + 24$.
10. $6a^3 - 23a^2 + 26a - 8$ and $2a^3 - 13a^2 + 27a - 18$.
11. $5x^4 + 10x^3 + x + 2$ and $10x^4 - 15x^3 + 2x - 3$.
12. $p^4 - p^3 + 3p^2 - 2p + 20$ and $2p^4 + p^3 + 6p^2 - 4p + 16$.
13. $x^6 + 3x^5 + 2x + 6$ and $x^6 + 2x^5 + 2x + 4$.

Exercise 36. Review

Factor the following expressions:

1. $x^2 - 4x - 21$.
2. $2x^2 - 11x - 21$.
3. $6x^2 - 5x - 21$.
4. $6x^2 - 23x + 21$.
5. $6x^2 - 27x + 21$.
6. $x^3 - 6x^2 - 13x + 42$.
7. $2x^3 - 13x^2 - 10x + 21$.
8. $6x^3 + x^2 - 26x - 21$.
9. $6x^3 - 29x^2 + 44x - 21$.
10. $6x^3 - 9x^2 - 60x + 63$.

Solve the following equations:

11. $x^2 - 26x + 165 = 0$.
12. $x^2 - 24x + 143 = 0$.
13. $x^2 - 33x + 260 = 0$.
14. $x^2 + 26x + 165 = 0$.
15. $x^2 + 23x - 50 = 0$.
16. $x^3 - 8x^2 + 19x - 12 = 0$.
17. $x^3 - 4x^2 - 9x + 36 = 0$.
18. $4x^2 - 16x + 15 = 0$.
19. $3x^2 + 8x = 35$.
20. $35x^2 - 9x = 2$.

Find the H.C.F. of the following expressions:

21. $a^2 + b^2 - 2ab$, $a^3 - b^3 + 3ab(b - a)$.
22. $a^3 + b^3$, $a^2 - b^2$, $a^7 + a^6b$, $a^5 + b^5$.
23. $a^4 - b^4$, $a^3 - b^3$, $a^3 + 2a^2b - ab^2 - 2b^3$.
24. $x^3 + x^2 - 14x - 24$, $x^3 - 19x - 30$, $x(x + 5) + 6$.
25. $x^3 - 13x - 12$, $x^3 + x^2 - 14x - 24$, $x^2 - x - 12$.

Find the L.C.M. of the following expressions:

26. 12, 16, 36.
27. 25, 35, 75.
28. $45p^3(x + y)$, $90q^3(x^2 - y^2)$.
29. $a^3b^3 - 1$, $a^2b^2 - 1$.
30. $a^2 - b^2$, $a^6 - a^2b^4$.
31. $a^3 - b^3$, $a^3 + b^3$.

Factor the following expressions:

32. $(a + b)^3 + (x + y)^3 + 3(a + b)(x + y)(a + b + x + y)$.
33. $(a + b + c)^3 + 8 + 6(a + b + c)^2 + 12(a + b + c)$.
34. $(a - b + c)^3 - d^3 - 3(a - b + c)^2d + 3(ad^2 - bd^2 + cd^2)$.

CHAPTER III

FRACTIONS

53. Reduction of Fractions. We reduce algebraic fractions to fractions having lower terms just as in arithmetic, by dividing both terms by their common factors. Similarly, we reduce to fractions having higher terms by multiplying both terms by the same quantity.

Just as $\frac{10}{15}=\frac{2}{3}$, so $\frac{a+b}{a^2-b^2}=\frac{a+b}{(a+b)(a-b)}=\frac{1}{a-b}$.

Multiplying or dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.

To reduce a fraction to lowest terms, divide both numerator and denominator by all their common factors.

We might divide by the H.C.F., but since we can usually find the H.C.F. by factoring, it is easier to divide by the factors as we discover them.

Reduce $\frac{x^3+5x^2+8x+6}{x^3-2x-4}$ to lowest terms.

Since it is difficult to factor numerator and denominator, we resort to a simple and convenient device.

Since a factor of each of two quantities is a factor of their difference (§ 49), we subtract the denominator from the numerator. The result is $5x^2+10x+10$.

Hence if there is any common factor, it is contained in $5x^2+10x+10$, or $5(x^2+2x+2)$. Evidently 5 is not a common factor. Trying x^2+2x+2, we find that it is a common factor. Dividing by it, we have

$$\frac{x^3+5x^2+8x+6}{x^3-2x-4}=\frac{(x+3)\cancel{(x^2+2x+2)}}{(x-2)\cancel{(x^2+2x+2)}}=\frac{x+3}{x-2}.$$

54. Changing Signs in the Terms. Since, from the law of signs in division (§ 19),

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b},$$

we see that

The value of a fraction is not altered by changing the signs of the numerator and of the denominator; by changing the signs of the fraction and of the numerator; or by changing the signs of the fraction and of the denominator.

It must be remembered that to change the sign of the numerator means that we must change the sign of *every term* of the numerator.

Exercise 37. Reduction of Fractions to Lowest Terms

Reduce the following to lowest terms:

1. $\frac{x^2+5x+6}{x^2+6x+8}$.
2. $\frac{a^2+a-6}{8-2a-a^2}$.
3. $\frac{a^2-9}{12-a-a^2}$.
4. $\frac{x^2+2x-35}{40-3x-x^2}$.
5. $\frac{6x^2-5x-4}{6x^2+x-12}$.
6. $\frac{6x^2-5x-4}{12-x-6x^2}$.
7. $\frac{2x^2+13x+21}{2x^2+10x+12}$.
8. $\frac{3x^2-2x-21}{3x^2+19x+28}$.
9. $\frac{4x^2-7x+3}{5x^2-3x-2}$.
10. $\frac{3x^3+5x^2+4x+2}{3x^3-x^2-2}$.
11. $\frac{4a^3+13a^2+4a+3}{4a^3+11a^2-2a+3}$.
12. $\frac{4a^3+13a^2+4a+3}{4a^3-11a^2-2a-3}$.
13. $\frac{5a^3-4a^2+a-2}{5a^3+6a^2+3a+2}$.
14. $\frac{3a^3+2a^2-a+14}{3a^3+4a^2-3a+2}$.
15. $\frac{a^3-6a^2+11a-6}{7a-a^3-6}$.
16. $\frac{2a^3+14a^2+33a+21}{2a^3+17a^2+27a+12}$.
17. $\frac{3a^3+22a^2+47a+28}{23a+21-3a^3-a^2}$.
18. $\frac{a^3-6a^2-6a+1}{a^3-2a^2-34a+5}$.

55. Equivalent Fractions. When two fractions are such that either may be obtained from the other by multiplying or dividing both of its terms by the same expression, they are called *equivalent fractions.*

56. Finding the Lowest Common Denominator. When several fractions are given in their lowest terms, we reduce any one of them to an equivalent fraction having any required denominator by multiplying both terms by the same expression. Therefore any common denominator of several fractions must be a common multiple of the given denominators. Therefore

The lowest common denominator of several fractions is the lowest common multiple of their denominators.

1. Reduce $\frac{2\,a^2b}{3\,x^2y}$, $\frac{3\,ab^2}{2\,xy^2}$, and $\frac{ab}{xy}$ to equivalent fractions having the lowest common denominator.

The L.C.D. is $6x^2y^2$.

Dividing $6x^2y^2$ by the respective denominators, we have $2y$, $3x$, $6xy$.

Multiplying both terms of the fractions by these factors, we have

$$\frac{4\,a^2by}{6\,x^2y^2}, \quad \frac{9\,ab^2x}{6\,x^2y^2}, \quad \frac{6\,abxy}{6\,x^2y^2}.$$

Fractions should be expressed in lowest terms before beginning to find the L.C.D.

2. Reduce $\frac{x-y}{x^2+3\,xy+2\,y^2}$ and $\frac{x}{2\,x^2+3\,xy+y^2}$ to equivalent fractions having the lowest common denominator.

The L.C.D. must contain the two denominators

$$x^2+3\,xy+2\,y^2=(x+y)(x+2\,y)$$

and $$2\,x^2+3\,xy+y^2=(x+y)(2\,x+y).$$

Therefore the L.C.D. $=(x+y)(x+2\,y)(2\,x+y)$.

Dividing, the multiplying factors are $2x+y$ and $x+2y$, respectively.

Therefore $$\frac{x-y}{(x+y)(x+2\,y)}=\frac{(x-y)(2\,x+y)}{(x+y)(x+2\,y)(2\,x+y)}$$

$$\frac{x}{(x+y)(2\,x+y)}=\frac{x(x+2\,y)}{(x+y)(x+2\,y)(2\,x+y)}.$$

Exercise 38. Reducing Fractions to the Lowest Common Denominator

Express with lowest common denominator:

1. $\frac{x+y}{x-y}, \frac{x+y}{x^2-y^2}$.
2. $\frac{m^2+n^2}{m^2-n^2}, \frac{m+n}{m-n}$.
3. $\frac{p-q}{p^5+q^5}, \frac{p-q}{p+q}$.
4. $\frac{m}{m+n}, \frac{n}{m^2-n^2}$.
5. $\frac{p}{p-1}, \frac{q}{p^2+1-2p}$.
6. $\frac{m}{b^2+c^2}, \frac{n}{b^2-c^2}$.

7. Express $\frac{x-y}{x+y}$ and $\frac{x+y}{x-y}$ with lowest common denominator. Then let $x=6$ and $y=4$, and show that, although the result obtained is correct, the *lowest* common denominator is not always the *least* common denominator.

Express with lowest common denominator:

8. $\frac{x}{x^2+y^2}, \frac{y}{x^2-y^2}$.
9. $\frac{x^2}{x+5}, \frac{x}{x+7}$.
10. $\frac{a-b}{a+b}, \frac{a^2-b^2}{a^2+2ab+b^2}$.
11. $\frac{4p^2-1}{4p^2+1}, \frac{4p^2+1}{4p^2-1}$.
12. $\frac{a^2}{a^3+b^3}, \frac{b^2}{a^2-ab+b^2}$.
13. $\frac{a^2+b^2}{a^2-b^2}, \frac{a^4+b^4}{a^4-b^4}$.
14. $\frac{x}{x^2-y^2}, \frac{y}{x^3+y^3}$.
15. $\frac{1}{a^3-b^3}, \frac{1}{a^6-b^6}$.
16. $\frac{1}{p^4-q^4}, \frac{1}{q^2-p^2}$.
17. $\frac{1}{a^4-b^4}, \frac{1}{a^8-b^8}$.
18. $\frac{a}{x+y}, \frac{b}{x-y}, \frac{c}{x^2-y^2}$.
19. $\frac{a}{a-b}, \frac{b}{a^2-b^2}, \frac{c}{a+b}$.
20. $\frac{a}{a-b}, \frac{b}{b-c}, \frac{c}{c-a}$.
21. $\frac{a+2}{a-2}, \frac{a^2+4}{a^2-4}, \frac{a^3+8}{a^3-8}$.
22. $\frac{1}{y^2+z^2}, \frac{1}{z^2+x^2}, \frac{1}{x^2+y^2}$.
23. $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2+b^2+c^2}$.
24. $\frac{a}{(a+b)(b+c)}, \frac{b}{(c+b)(a+c)}$.
25. $\frac{1}{(p^2+q^2)(p^2-q^2)}, \frac{1}{p+q}$.

57. Finding the Algebraic Sum of Fractions. Therefore, to find the algebraic sum of several fractions we proceed as follows:

If the fractions have the same denominators, write the algebraic sum of the numerators over the common denominator.

If the fractions have different denominators, express them as equivalent fractions having the lowest common denominator, and write the algebraic sum of the new numerators over the common denominator.

In either case, reduce the resulting fraction to lowest terms.

Since we are considering the algebraic sum, this case covers both addition and subtraction.

Exercise 39. Addition and Subtraction

Add or subtract as indicated:

1. $\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}$.

2. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{ab}+\frac{1}{bc}$.

3. $\frac{1}{a+b}+\frac{1}{a-b}+\frac{1}{a^2-b^2}$.

4. $\frac{1}{4}+\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a-b}$.

5. $\frac{a^2-2ab+b^2}{(a+b)^2}+\frac{(a+b)^2}{a^2-2ab+b^2}$.

6. $\frac{a+b}{a^2+ab+b^2}+\frac{a-b}{a^2-ab+b^2}$.

7. $\frac{2a-b}{4a^2-2ab+b^2}+\frac{2a+b}{4a^2+2ab+b^2}$.

8. $\frac{x^2+2xy+y^2}{x^2-2xy+y^2}-\frac{x^2-2xy+y^2}{x^2+2xy+y^2}-\frac{8xy^3}{(x^2-y^2)^2}$.

9. $\frac{m+2}{m^2+m}+\frac{m-2}{m}-\frac{m}{m^2+2m+1}$

10. $\frac{a+b}{x-y}+\frac{a-b}{x+y}+\frac{a^2-b^2}{x^2-y^2}$.

11. $\frac{a^2+ab+b^2}{a^2-ab+b^2}-\frac{a^2-ab+b^2}{a^2+ab+b^2}$.

12. $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}$.

13. $\frac{a+b}{a-b}+\frac{b+c}{b-c}+\frac{c+a}{c-a}$.

58. Special Suggestions in Addition. The problems already solved are typical of most of the problems that will be met in the addition of fractions. A few other forms are occasionally found, and three of these types will now be considered.

1. Add $\frac{a}{b}$, $-\frac{a}{3b-2}$, and $\frac{2b+1}{4-9b^2}$.

Since $4-9b^2=(2+3b)(2-3b)$, we change the sign of the second fraction and of its denominator (§ 54).

$$\frac{a}{b}+\frac{a}{2-3b}+\frac{2b+1}{4-9b^2}=\frac{a(4-9b^2)}{b(4-9b^2)}+\frac{ab(2+3b)}{b(4-9b^2)}+\frac{b(2b+1)}{b(4-9b^2)}$$
$$=\frac{4a-6ab^2+2ab+2b^2+b}{b(4-9b^2)}.$$

2. Find the value of $\frac{1}{x^2+3x-10}-\frac{1}{6-5x+x^2}$.

$x^2+3x-10=(x+5)(x-2)$.

$6-5x+x^2=(3-x)(2-x)$, or $(x-3)(x-2)$, by § 45.

We therefore have

$$\frac{1}{(x+5)(x-2)}-\frac{1}{(x-3)(x-2)}=\frac{-8}{(x+5)(x-2)(x-3)}.$$

3. Find the value of

$$\frac{1}{(x-y)(y-z)}+\frac{1}{(z-y)(x-z)}+\frac{1}{(z-x)(y-x)}.$$

We see that $y-z$ is the same as $z-y$ except for its signs, and similarly for $x-y$ and $y-x$, and for $x-z$ and $z-x$.

But in the second fraction $(z-y)(x-z)=(y-z)(z-x)$, by § 45.

In the third fraction $(z-x)(y-x)=-(z-x)(x-y)$.

We may therefore write the fractions thus:

$$\frac{1}{(x-y)(y-z)}+\frac{1}{(y-z)(z-x)}-\frac{1}{(z-x)(x-y)}.$$

These equal

$$\frac{z-x}{(x-y)(y-z)(z-x)}+\frac{x-y}{(x-y)(y-z)(z-x)}-\frac{y-z}{(x-y)(y-z)(z-x)}$$
$$=\frac{2z-2y}{(x-y)(y-z)(z-x)}=\frac{-2}{(x-y)(z-x)}.$$

Exercise 40. Addition and Subtraction of Fractions

Examples 1 to 8, oral — Examples 9 to 17, written

1. What is the sum of $\frac{1}{a}$ and $\frac{1}{-a}$?

2. Add $\frac{1}{a}$, $\frac{1}{-a}$, $\frac{-1}{a}$, and $\frac{-1}{-a}$.

Add or subtract as indicated:

3. $\frac{2}{a}+\frac{1}{-a}$.

4. $\frac{5}{a^2}-\frac{1}{-a^2}$.

5. $\frac{1}{xy}-\frac{2}{-xy}$.

6. $\frac{3}{mn}-\frac{4}{-mn}$.

7. $\frac{1}{a-b}+\frac{1}{b-a}$.

8. $\frac{3}{x-4}+\frac{1}{4-x}$.

9. Find the value of $\frac{1}{(x-2)(x-3)}+\frac{1}{(x-1)(2-x)}$, and check the result by letting $x=4$.

Add or subtract as indicated:

10. $\frac{1}{(a-b)(a+b)}+\frac{2}{(b-a)(a+b)}+\frac{3}{b^2-a^2}$.

11. $\frac{4a}{(a-5)(a-b)}-\frac{3a}{(5-a)(a-b)}$.

12. $\frac{2}{a^2-10a+21}+\frac{2}{a-7}-\frac{3}{3-a}$.

13. $\frac{5}{x^2-14x+45}-\frac{3}{x-5}+\frac{2}{9-x}$.

14. $\frac{1}{(x-1)(x-2)}+\frac{2}{(x-2)(3-x)}-\frac{3}{(x-3)(1-x)}$.

15. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$.

16. $\frac{3}{(p-q)(q-r)}-\frac{4}{(r-q)(p-r)}+\frac{2}{(r-p)(p-q)}$.

17. $\frac{a}{(x-y)(y-z)}+\frac{b}{(z-x)(y-x)}+\frac{c}{(y-z)(x-z)}$.

59. Multiplication of Fractions. We multiply fractions in algebra in the same manner as in arithmetic. Just as $5 \times \frac{2}{3} = \frac{10}{3}$, so $a \cdot \frac{b}{c} = \frac{ab}{c}$. Therefore

To multiply a fraction by an integral expression, multiply the numerator, writing the product over the denominator.

Similarly, just as $\frac{2}{3}$ of $\frac{4}{5}$ equals $\frac{2 \times 4}{3 \times 5}$, or $\frac{8}{15}$, so $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Therefore

To multiply a fraction by a fraction, multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Cancel factors common to any numerator and any denominator before multiplying.

1. Find the product of $\frac{x^2y}{2m^2n} \times 4 \times \frac{5mn^3}{6xy^4} \times \frac{3y^2}{5m}$.

Indicating the multiplication, we have

$$\frac{3 \cdot 4 \cdot 5 \cdot mn^3x^2y^3}{2 \cdot 5 \cdot 6 \cdot m^3nxy^4}.$$

Canceling common factors, we have $\frac{n^2x}{m^2y}$.

2. Find the product of $\frac{x^2-x-6}{x^2+x-6} \cdot \frac{x^2+2x-8}{x^2-2x-8} \cdot \frac{x^2-x-12}{x^2+x-12}$.

Factoring, for ease in canceling, and then canceling common factors, we have

$$\frac{\cancel{(x+2)}\cancel{(x-3)}}{\cancel{(x-2)}\cancel{(x+3)}} \cdot \frac{\cancel{(x-2)}\cancel{(x+4)}}{\cancel{(x+2)}\cancel{(x-4)}} \cdot \frac{\cancel{(x+3)}\cancel{(x-4)}}{\cancel{(x-3)}\cancel{(x+4)}} = 1.$$

3. Simplify $\left(x - 1 - \frac{4}{x-1}\right)\left(x^3 + 3x^2 + 8x + 24 + \frac{72}{x-3}\right)$.

Reducing to fractional forms,

$$\frac{x^2-2x-3}{x-1} \cdot \frac{x^4-x^2}{x-3} = \frac{\cancel{(x-3)}(x+1)}{\cancel{x-1}} \cdot \frac{x^2\cancel{(x+1)}\cancel{(x-1)}}{\cancel{x-3}} = x^2(x+1)^2.$$

Check. Let $x=2$. Then $(2-1-4)(8+12+16+24-72) = -3 \cdot -12 = 36$, and $4 \cdot 9 = 36$.

Exercise 41. Multiplication of Fractions

Multiply as indicated:

1. $\frac{a+b}{a-b} \cdot \frac{a-b}{a+b}$.

2. $\frac{a+b}{a^2+b^2} \cdot \frac{a^2+b^2}{a^2-b^2}$.

3. $\frac{x+2}{x^2+4} \cdot \frac{x^2+4}{x^2-4}$.

4. $\frac{2}{a-b} \cdot \frac{a^2-b^2}{6}$.

5. $\frac{x^3}{x^2-y^2} \cdot \frac{x+y}{x^4y^4}$.

6. $\frac{x^3-y^3}{x^3+y^3} \cdot \frac{x+y}{x-y}$.

7. $\frac{a(a-b)}{a^2+2ab+b^2} \cdot \frac{b(b+a)}{b^2-2ba+a^2} \cdot \frac{a^2-b^2}{ab}$.

8. $\frac{x^3+y^3}{x^3-y^3} \cdot \frac{x^2+y^2}{x^2-y^2} \cdot \frac{x+y}{x-y} \cdot \frac{y^2-x^2}{y^2+x^2}$.

9. $\frac{m^3-n^3}{m^2-9} \cdot \frac{3-m}{m+n} \cdot \frac{3+m}{m^2-mn+n^2}$.

10. $\frac{p^3-q^3}{p^3+q^3} \cdot \frac{p+q}{p-q} \cdot \frac{p^2-pq+q^2}{p^2+pq+q^2}$.

11. $\frac{a^2+2ab+b^2}{a-b} \cdot \frac{a^2-2ab+b^2}{a+b} \cdot \frac{a-b}{a+b}$.

12. $\frac{x^2-y^2}{p-q} \cdot \frac{q^2-p^2}{y-x} \cdot \frac{3}{p+q} \cdot \frac{3}{x-y}$.

13. $\frac{a+b-c}{x} \cdot \frac{a-b+c}{y} \cdot \frac{x+y-z}{a} \cdot \frac{x-y+z}{b}$.

14. $\left(\frac{a}{m}-\frac{m}{a}+\frac{n}{b}-\frac{b}{n}\right)\left(\frac{a}{m}-\frac{m}{a}-\frac{n}{b}+\frac{b}{n}\right)$.

15. $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$.

16. $\frac{x^2-5x+6}{x^2-9x+20} \cdot \frac{x^2-11x+28}{x^2-8x+12} \cdot \frac{x^2-11x+30}{x^2-10x+21}$.

17. $\frac{x^2+3x+2}{x^2+8x+15} \cdot \frac{x^2+7x+12}{x^2-5x-6} \cdot \frac{x^2-x-30}{x^2+6x+8}$.

18. $\frac{x^2-1}{x^2+5x+6} \cdot \frac{x^2-4}{x^2-2x-3} \cdot \frac{x^2-9}{x^2-3x+2}$.

60. Reciprocal. If the product of two quantities is 1, each is called the *reciprocal* of the other.

Since $\frac{a}{b} \cdot \frac{b}{a} = 1$, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, and conversely.

61. Division of Fractions. Consider the following cases:

$\$2 \div \$3 = \frac{2}{3}$, the "dollars" disappearing in the quotient.

2 ft. $\div$ 3 ft. $= \frac{2}{3}$, the denomination "feet" disappearing.

$\frac{2}{5} \div \frac{3}{5} = \frac{2}{3}$, the denomination "fifths" disappearing.

To divide $\frac{a}{b}$ by $\frac{c}{d}$ is the same as to divide $\frac{ad}{bd}$ by $\frac{bc}{bd}$ (§ 53).

$\frac{ad}{bd} \div \frac{bc}{bd} = \frac{ad}{bc}$, the denomination "$bd$ths" disappearing.

But $\frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$. Therefore $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. Therefore

To divide by a fraction, multiply by its reciprocal.

Exercise 42. Division of Fractions

Divide as indicated:

1. $\frac{105\,ab}{64\,xy} \div \frac{15\,a^2b^2}{24\,x^2y^2}$.

2. $\frac{19x^2yz}{25\,m^3n^2} \div \frac{57\,xy^2z^3}{85\,m^4n^5}$.

3. $\frac{30\,a^4b^5c^6}{23x^4y^5z^6} \div \frac{25\,a^3b^2c}{69x^2y^3z^4}$.

4. $\frac{25\,m^5n^4z}{66\,a^4b^5c^6} \div \frac{35\,m^3nz^4}{88\,a^3b^3c^3}$.

5. $\frac{a^6+b^6}{a^2-b^2} \div \frac{a^6-b^6}{a^2+b^2}$.

6. $\frac{a^6+b^6}{a^2-b^2} \div \frac{a^2+b^2}{a^6-b^6}$.

7. $\frac{a^8-b^8}{a^8+b^8} \div \frac{a^4-b^4}{a^4+b^4}$.

8. $\frac{x^2+y^2}{x^2-y^2} \div \frac{5x^2+5y^2}{x-y}$.

9. $\frac{4(a^2+b^2-c^2+2ab)}{c^2-a^2-b^2+2ab} \div \frac{2(a+b+c)}{b+c-a}$.

10. $\frac{a^3+8b^3+6ab(a+2b)}{a^3-8b^3-6ab(a-2b)} \div \frac{a(a+4b)+4b^2}{a(a-4b)+4b^2}$.

11. $\left(\frac{1+m}{1-m} - \frac{1-n}{1+n}\right) \div \left[1 + \frac{(1+m)(1-n)}{(1-m)(1+n)}\right] \div (1+m)$.

62. The Laws of Fractions. If desired, the usual laws of fractions may be presented with less regard to arithmetic, as follows:

1. *Multiplying both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

If $\frac{a}{b} = x,$

then $a = bx,$ Why?

and $na = nbx.$

Therefore $\frac{na}{nb} = x.$ Why?

But $\frac{a}{b} = x,$

and hence $\frac{a}{b} = \frac{na}{nb}.$ Why?

2. *Dividing both numerator and denominator of a fraction by the same expression does not change the value of the fraction.*

For it has just been proved that

$$\frac{na}{nb} = \frac{a}{b}.$$

3. *Multiplying the numerator by any number multiplies the fraction by that number, and dividing the numerator by any number divides the fraction by that number.*

Let x be the value of the fraction $\frac{a}{b}$.

Then $\frac{a}{b} = x,$

and $a = bx,$

and $na = nbx.$

Then $\frac{na}{b} = nx.$ Why?

That is, multiplying a by n multiplies x, the value of the fraction, by n.

Likewise, since $a = bx,$

therefore $a \div n = bx \div n,$

and $\frac{a \div n}{b} = \frac{bx \div n}{b} = x \div n.$

That is, dividing a by n divides x, the value of the fraction, by n.

4. *Multiplying the denominator by any number divides the fraction by that number, and dividing the denominator by any number multiplies the fraction by that number.*

Let $\frac{a}{b} = x.$

Then $a = bx,$

and $a = \frac{nbx}{n}.$ Law 1

Therefore $\frac{a}{nb} = \frac{x}{n}.$ Why?

That is, multiplying b by n divides x, the value of the fraction, by n. The proof for division by n is substantially the same.

5. *To multiply one fraction by another, take the product of the numerators for a numerator, and the product of the denominators for a denominator.*

Let the two fractions be $\frac{a}{b}$ and $\frac{m}{n}$.

Let $\frac{a}{b} = x,$ whence $a = bx;$

and let $\frac{m}{n} = y,$ whence $m = ny.$

Then, multiplying, $am = bnxy.$ Why?

Hence, dividing, $\frac{am}{bn} = xy.$

That is, xy, which is the product of the fractions, is found by taking am for the numerator and bn for the denominator.

6. *To divide one fraction by another, multiply the dividend by the reciprocal of the divisor.*

Let $\frac{a}{b} \div \frac{m}{n} = x.$

Then $\frac{a}{b} = x \cdot \frac{m}{n} = \frac{xm}{n}.$ Why?

Then $\frac{an}{b} = xm,$

and $\frac{an}{bm} = x.$ Why?

But $\frac{an}{bm}$ is the same as $\frac{a}{b}$ multiplied by the reciprocal of $\frac{m}{n}$.

63. Complex Fraction. A fraction that has one or more fractions in either or both of its terms is called a *complex fraction.*

A complex fraction is merely an indicated case of the division of fractions. Thus $\dfrac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}$ simply indicates that $1+\dfrac{1}{x-1}$ is to be divided by $1-\dfrac{1}{x+1}$.

Reducing both terms of the complex fraction to fractional forms and simplifying, we have a fraction in which we may multiply both terms by $(x-1)(x+1)$, or which we can treat as a case of division, thus:

$$\frac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}=\frac{\dfrac{x}{x-1}}{\dfrac{x}{x+1}}=\frac{x(x+1)}{x(x-1)}=\frac{x+1}{x-1}.$$

1. Simplify $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x^2+y^2}{x^2-y^2}}$.

Multiply both numerator and denominator of the complex fraction by x^2-y^2, which is the L.C.D. of the simple fractions. This reduces the complex fraction to

$$\frac{(x+y)^2+(x-y)^2}{x^2+y^2}=\frac{x^2+2xy+y^2+x^2-2xy+y^2}{x^2+y^2}$$
$$=\frac{2x^2+2y^2}{x^2+y^2}=2.$$

2. Simplify $\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}$.

Multiply both terms of the last complex fraction by x. This gives $\dfrac{x}{x^2+1}$, the complex fraction then reducing to $\dfrac{1}{x+\dfrac{x}{x^2+1}}$. Multiplying both terms of this fraction by x^2+1, we have $\dfrac{x^2+1}{x^3+2x}$.

A complex fraction of this form is also called a *continued fraction.*

Exercise 43. Complex Fractions

Examples 1 to 5, oral — Examples 6 to 17, written

1. Simplify $\frac{1+\frac{1}{2}}{1-\frac{1}{2}}$; $\frac{1+\frac{1}{4}}{1-\frac{1}{4}}$; $\frac{1-\frac{1}{5}}{1+\frac{1}{5}}$; $\frac{1-\frac{1}{6}}{1+\frac{1}{6}}$; $\frac{\frac{1}{4}}{1+\frac{1}{4}}$.

2. Simplify $\frac{\frac{1}{2}}{1+\frac{1}{2}}$; $\frac{\frac{2}{3}}{1-\frac{2}{3}}$; $\frac{\frac{3}{3}}{1+\frac{1}{3}}$; $\frac{\frac{3}{4}}{1-\frac{3}{4}}$; $\frac{\frac{3}{4}}{1+\frac{3}{4}}$.

Simplify the following:

3. $\dfrac{\frac{a+b}{a}}{\frac{a-b}{b}}$.

4. $\dfrac{a+\frac{a}{b}}{a-\frac{a}{b}}$.

5. $\dfrac{x+\frac{1}{y}}{x-\frac{1}{y}}$.

6. $\dfrac{\frac{1+a}{a}}{1-\frac{1}{a^2}}$.

7. $\dfrac{\frac{a+b}{a-b}}{\frac{a-b}{a+b}}$.

8. $\dfrac{\frac{x-y}{x+y}}{\frac{x^2-y^2}{x^2+y^2}}$.

9. $\dfrac{\frac{a^3}{b^3}+1}{1+\frac{a}{b}}$.

10. $\dfrac{16-\frac{a^4}{b^4}}{\frac{a}{b}+2}$.

11. $\dfrac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{\frac{a-b}{a+b}+\frac{a+b}{a-b}}$.

12. $\dfrac{x-1+\frac{y}{x-y}}{x-2+\frac{y}{x+y}}$.

13. $\dfrac{a}{1+\frac{b}{c+\frac{1}{d}}}$.

14. $\dfrac{\frac{p+q}{p}-\frac{p-q}{q}}{\frac{p+q}{q}+\frac{p-q}{p}}$.

15. $\dfrac{a-b+\frac{c}{a+b}}{a+b-\frac{c}{a-b}}$.

16. $\dfrac{x}{y+\frac{1}{y+\frac{1}{x}}}$.

17. Simplify $\dfrac{m+\frac{mn}{m-n}}{m-\frac{mn}{m-n}}$, and check the result.

CHAPTER IV

FUNCTIONS

64. Function. An algebraic expression that depends upon another for its value is called a *function* of the latter.

For example, the formula for the circumference of a circle is $c = 2\pi r$. Here the factors 2 and π are definite numbers, but r may have any value, and the value of c depends upon the value we give to r. Therefore c is called a *function* of r.

Similarly, in the area of a circle we have $a = \pi r^2$. Here a depends upon the value of r, and therefore is a function of r.

In computing interest we may use the formula $i = prt$. Here i is a function of p, a function of r, and also a function of t.

65. Symbols. If some function of r is to be referred to several times in a discussion, it is convenient to have some symbol to represent it. It is customary to use $f(r)$ as this symbol, and to read it "function of r," or simply "f of r."

Thus if we are going to speak a number of times of the expression $x^3 + 3x^2 + 3x + 1$, it is convenient to let $f(x) = x^3 + 3x^2 + 3x + 1$ and to speak of it always as $f(x)$ in the discussion. If we have two different expressions that are functions of x, we may speak of one as $f(x)$ and the other as $F(x)$, — "f minor of x" and "f major of x," respectively.

If we wish to represent a function of a and b we write it $f(a, b)$, reading this "f of a, b," and similarly for three or more quantities.

A series of dots is read "and so on," as in $x = 1, 2, 3, \cdots$, to 10.

66. Evaluation of Functions. To *evaluate* $f(x)$ is to put the given value of x in place of x in the function.

Thus if $f(x) = x^2 - x + 9$, we evaluate it for $x = 3$ by putting 3 in place of x. Then $3^2 - 3 + 9 = 15$. This is usually expressed thus:

If $\quad f(x) = x^2 - x + 9,$

then $\quad f(3) = 3^2 - 3 + 9 = 15.$

That is, $f(3)$ means the value of $f(x)$ when 3 is put in place of x.

67. Variable. A quantity that, under the conditions of a problem, may take different values is called a *variable.*

In the equation $y = 4x - 2$ we may give to x various values and from these we may find corresponding values for y. This equation has therefore two variables, x and y.

68. Constant. A quantity that, under the conditions of a problem, has a fixed value is called a *constant.*

In the formula $c = 2\pi r$, r may vary, in which case c will also vary; but 2 and π are constants.

69. Dependent and Independent Variable. When a formula explicitly states the value of one variable in terms of one or more others, we think of the former as depending upon the latter for its value. In this case we call the former a *dependent variable* and the latter an *independent variable* or *independent variables.*

In the formula $c = \pi d$, c is the dependent variable and d is the independent variable. If we derive from it the formula $d = \frac{c}{\pi}$, then d is the dependent variable and c is the independent variable.

70. Importance of the Function Idea. The idea of the function is very important. Every formula represents some quantity as a function of one or more other quantities.

One of the aims of mathematics is to lead us to think "functionally," to see the dependence of one quantity upon another. Indeed, to solve the equation $x^2 + px + q = 0$ is to express x as a function of the coefficients, as is the case when we write $x = -\frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$.

Exercise 44. Evaluation of Functions

Examples 1 to 5, oral — Examples 6 to 20, written

1. If $f(x) = x + 9$, what is $f(2)$? $f(3)$? $f(-9)$?
2. If $f(x) = x^2 - 1$, what is $f(2)$? $f(4)$? $f(5)$? $f(7)$?
3. If $f(x) = x^2 + x + 1$, what is $f(1)$? $f(2)$? $f(3)$? $f(5)$?
4. If $f(r) = \frac{4}{7}r$, what is $f(7)$? $f(1)$? $f(14)$? $f(\frac{1}{7})$?
5. If $f(p) = 6\%\, p$, what is $f(100)$? $f(1000)$? $f(2000)$?

Evaluate the following functions:

6. $f(m) = 2m + 5$, for $m = 1.9$; for $m = 26$.

7. $f(x) = x^2 - x + 7$, for $x = 3$; for $x = 4$.

8. $f(y) = 2y^2 + y - 8$, for $y = 4$; for $y = 5$.

9. $f(m) = m^3 + m^2 + m + 1$, for $m = 1$; for $m = 2$.

10. $f(a) = a^3 + 3a^2 + 3a + 1$, for $a = 3$; for $a = 4$; for $a = 10$.

11. If $f(x) = x^2 + 2x + 1$, what is $f(2)$? $f(5)$? $f(7)$? $f(11)$?

12. If $f(x) = x^2 + x + 1$ and $F(x) = x^2 - x + 1$, find the value of $f(3)$ and $F(4)$, and then of $f(3) + F(4)$.

13. If $f(x) = 4x^2 + 1$, and $F(x) = 7x^2 + 2x + 3$, find the value of $f(5)$ and $F(2)$, and then of $f(5) + F(2)$.

14. If $f(x) = x^2 - 4x + 4$, which is greater, $f(2)$ or $f(1)$? $f(1)$ or $f(0)$? $f(1)$ or $f(3)$? $f(10)$ or $f(9)$?

15. If $f(x) = x^2 - 6x + 9$, which is greater, $f(0)$ or $f(2)$? $f(1)$ or $f(5)$? $f(2)$ or $f(4)$? $f(6)$ or $f(0)$?

16. If it is known that a rectangle is 10 in. long and h inches high, the area is a function of what quantity? What is the area when $h = 1$? 3? 9? $5\frac{1}{2}$? 7.4? 0?

17. If it is known that a triangle is 8 in. high and has a base b inches, the area is a function of what quantity? What is the area when $b = 5$? 7? $9\frac{1}{4}$? 5.7? 0?

18. In the formula $c = 2\pi r$, c is a function of what letter? Give the various values of this function for $r = 0, 1, 2, 3, 4, 5$, using $3\frac{1}{7}$ for the value of π in this and similar cases.

19. If $f(r) = \pi r^2$, make a table of its values for the following:

$r =$	0	1	2	3	4	5	6	7	8	10
$f(r) =$										

20. Make a similar table showing the values of $f(x) = x^2 + 1$, for $x = 0, 1, 2, \cdots, 10$.

71. Graphic Representation of Functions. An important aid in the study of functions is the graph.

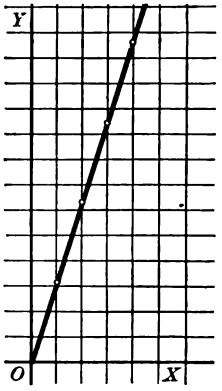
1. Study the rate of change of the circumference of a circle as the diameter changes.

We know (§ 4) that $c = \pi d$.

Let $d =$	0	1	2	3	4	5	6	7
Then $c =$	0	$3\frac{1}{7}$	$6\frac{2}{7}$	$9\frac{3}{7}$	$12\frac{4}{7}$	$15\frac{5}{7}$	$18\frac{6}{7}$	22

In this computation $\pi = 3\frac{1}{7}$.

If we take squared paper and mark off the values of d on the line OX, and the values of c upwards from OX, we shall find the series of points shown in the figure. Connecting these we have a straight line that represents the rate of change of c.



We therefore see that c changes at a uniform rate, and that as we double d we double c, and so on; and when we take d half as large, c will also be half as large. This we would also infer from the fact that $\frac{c}{d} = \pi$; that is, the ratio of c to d is constant.

2. Study the rate of change of the area of a circle as the diameter changes.

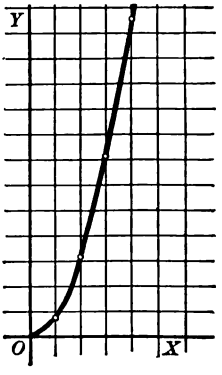
We know (§ 4) that $a = \pi r^2$, or $a = \frac{1}{4}\pi d^2$. We therefore have the following:

Let $d =$	0	1	2	3	4
Then $a =$	0	$\frac{11}{14}$	$\frac{22}{7}$	$\frac{99}{14}$	$\frac{88}{7}$

For the purposes of the graph we may write the values of a as follows: 0, $\frac{11}{14}$, $\frac{44}{14}$, $\frac{99}{14}$, $\frac{176}{14}$, $\cdots$, or 0, 1 −, 3 +, 7 +, $12\frac{1}{2}$ +, $\cdots$.

Making the graph, we see that the area is changing much more rapidly than the diameter.

If we compare Exs. 1 and 2, we see that, in general, the area is also changing much more rapidly than the circumference.



Exercise 45. Graphs of Functions

1. Draw a graph showing the rate of change of the area of a square as the side changes.

2. Given the formula $s = \pi d^2$, draw a graph showing the rate of change of the surface of a sphere as the diameter changes.

3. Given the formula $v = \frac{1}{6}\pi d^3$, draw a graph showing the rate of change of the volume of a sphere as the diameter changes.

4. Draw a graph showing the rate of change of the diameter of a sphere as the volume changes.

5. Given $f(x) = x - 3$, draw a graph showing the rate of change of $f(x)$ as x changes.

First complete this table:

If $x =$	0	1	2	3	4	5	6	7	8	9
Then $f(x) =$	-3	-2								

Then represent the values of x on the line OX, and the values of $f(x)$ above or below the line OX.

6. Given $f(x) = x^2 + 1$, draw a graph showing the rate of change of $f(x)$ as x changes.

7. Given $f(x) = x^2 + x$, draw a graph showing the rate of change of $f(x)$ as x changes.

Draw graphs showing the rate of change of $f(x)$ as x changes in the following functions:

8. $f(x) = x^3 - x$.

9. $f(x) = x^3$

10. $f(x) = x^3 - 2$.

11. $f(x) = x^3 + 2$.

12. $f(x) = x^3 + x^2$.

13. $f(x) = x^3 - x^2$.

14. $f(x) = x^3 + x^2 + x$.

15. $f(x) = x^3 + x^2 + x + 1$.

Exercise 46. Evaluating Functions

Examples 1 to 15, oral — Examples 16 to 40, written

1. If $a = 7$, what is the value of $a^2 + 11$?

2. If $f(n) = n^2$, what is the value of $f(n)$ when $n = 0$? when $n = 1$? when $n = \frac{1}{2}$? when $n = 5$? when $n = 10$?

3. If $f(x) = x^2 + x$, what is the value of $f(x)$ when $x = 0$? when $x = 1$? when $x = 10$?

If $a = 2$ and $b = 4$, find the value of the following functions of a and b:

4. $\frac{a}{b}$.
5. $\frac{b}{a}$.
6. $\frac{a^2}{b}$.
7. $\frac{a+b}{b}$.
8. $\frac{b-a}{b}$.
9. $\frac{b+a}{b-a}$.
10. $\frac{a^2+b}{8}$.
11. $\frac{a+b^2}{9}$.
12. $\frac{a^3-b}{4}$.
13. $\frac{2a}{b}$.
14. $\frac{3a}{a+b}$.
15. $\frac{a^3-b}{8}$.

If $a = 5$ and $b = 2$, find the value of the following functions:

16. $0.2a$.
17. $0.5b$.
18. $0.7ab$.
19. $0.1ab^2$.
20. $0.2ba^2$.
21. $0.3a^3b$.
22. $6a^2 + b^2$.
23. $6a^2 - b^2$.
24. $25a^3 + b^3$.
25. $7a^b$.
26. $7b^a$.
27. a^{b+1}.

If $a = 6$, $b = -4$, and $c = 2$, find the value of the following functions of a, b, and c:

28. $\frac{a+b}{c}$.
29. $\frac{a+c}{b}$.
30. $\frac{b+c}{a}$.
31. $\frac{a+b+c}{a}$.
32. $\frac{a+b+c}{b}$.
33. $\frac{a+b+c}{c}$.
34. $\frac{a^2+b^2}{c^2}$.
35. $\frac{a^2-b^2}{c^2}$.
36. $\frac{a^2+2ab+b^2}{c^4}$.

If $x = 2$, $y = 3$, and $z = -4$, find the value of the following:

37. $x + y + z$.
38. $3x + 2y + 5z$.
39. $3x^2 - 2y^2 - 5z^2$.
40. $2xy + 3yz + 4zx$.

Exercise 47. Representing Functions

Examples 1 to 5, oral — Examples 6 to 16, written

1. If a represents the greater of two numbers and b the less, how will you represent their sum? their difference?

2. Using the same letters as in Ex. 1, how will you represent the sum of the squares of two numbers? the difference of their squares? the product of their cubes?

3. How will you represent twice some number a? 1 more than twice the number? 1 less than twice the number?

4. How will you represent the square of the sum of two numbers? the square of the difference of two numbers?

5. Using r for radius and c for circumference, how will you state the product of the radius and circumference? half of this product?

Letting a, b, and c represent the three numbers, express algebraically the following functions of a, b, and c:

6. The sum of three numbers.

7. The sum of the squares of three numbers.

8. Five times the product of three numbers.

9. The square of the first of three numbers divided by the sum of the other two.

10. The sum of the first two numbers divided by three times the cube of the third.

If the three dimensions of a rectangular box are l, w, and h, express algebraically the following functions of l, w, and h:

11. The volume of the box.

12. The area of each side and of the bottom.

13. The sum of the four edges that run lengthwise.

14. The sum of the four edges that run crosswise.

15. The sum of the four vertical edges.

16. The sum of all twelve edges of the box.

Exercise 48. Evaluating Functions

Examples 1 to 5, oral — Examples 6 to 14, written

1. If $a = 7$ and $b = 9$, find the value of $f(a, b) = a^2 + b$.

2. If $m = 5$ and $n = 30$, find the value of $f(m, n) = n + m$.

3. If $x = 8$ and $y = -4$, find the value of $f(y) = 2y$; of $f(x, y) = 2y + x$; of $f(x, y) = y^2 + x$; of $f(x, y) = x - 2y$.

4. How do you represent a times one number plus b times the same number?

5. How do you represent a times one number, plus b times another number, plus c times another number?

Letting x, y, and z represent the three numbers, express algebraically the following functions of x, y, and z:

6. The sum of three numbers. Evaluate for $x = y = z = 7$.

7. The sum of the squares of three numbers. Evaluate for $x = 1$, $y = 2$, $z = -3$; for $x = 0.1$, $y = 0.2$, $z = -0.3$.

8. The sum of the cubes of three numbers. Evaluate for $x = 1$, $y = 3$, $z = 5$; for $x = 7$, $y = 9$, $z = 8$.

9. The product of three numbers. Evaluate for $x = 4$, $y = -3$, $z = 10$; for $x = 17$, $y = -9$, $z = 24$.

10. The product of the squares of three numbers. Evaluate for $x = 5$, $y = 6$, $z = -7$; for $x = 0.1$, $y = 0.6$, $z = -0.9$.

11. The sum of three numbers divided by their product. Evaluate for $x = 4$, $y = -6$, $z = 10$; for $x = 7$, $y = 9$, $z = -3$.

12. The sum of the first two numbers diminished by the third number. Evaluate for $x = -10$, $y = 35$, $z = 17$.

13. The first number diminished by the sum of the second and third numbers. Evaluate for $x = 72$, $y = 29$, $z = -16$.

14. If $f(x) = x^2 + 1$, and $F(x) = x^2 - 2x + 15$, find the value of $f(2)$ and $F(5)$. Then find the value of $f(2) \cdot F(5)$; of $F(5) + f(2)$.

CHAPTER V

SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY

72. Equation. An expression of equality between two numbers or quantities is called an *equation.*

For example, $x + 5 = 7$. This means that some number increased by 5 equals 7. Evidently this number is 2.

In this equation x is usually called the *unknown quantity.* It is really a number to be determined. Unknown quantities are usually represented by x, y, or z, or else by initial letters, such as p for pounds or d for dollars.

In this equation $x + 5$ is called the *first member* and 7 is called the *second member.*

Simple types of equations have already been reviewed in Chapter I.

73. Simple Equation. An equation involving the first power and no higher power of the unknown quantity is called a *simple equation.*

Such an equation is also said to be of the *first degree.* It is also called, particularly in higher mathematics, a *linear equation.*

74. Identity. An equation that is true for any value whatsoever of any letter is called an *identity.*

Thus $(a + b)^2 = a^2 + 2ab + b^2$. This is true if $a = 3$, $b = 4$, or if the letters have any other values whatsoever.

An equation involving only numbers, like $2 + 3 = 5$, is also called an identity.

An identity is sometimes expressed by the symbol $\equiv$, as in the case of $a + b \equiv b + a$, which is read "$a + b$ is identical to $b + a$."

Sometimes an equation is distinguished from an identity by being called an *equation of condition.* This means that it is an equation that is true only for certain values of the letter or letters involved. But all equations containing an unknown quantity are, in general, equations of condition, so the name is seldom used.

The above definitions, familiar from a study of Book I, are here repeated by way of review.

75. Satisfying an Equation. The quantity that, substituted for the unknown quantity, reduces an equation to an identity is said to *satisfy* the equation.

This quantity is the value of the unknown quantity. Thus if $x + 7 = 9$, then $x = 2$, and 2 satisfies the equation.

If $ax = a^2$, and we consider x as the unknown quantity, we see that $x = a$, and that a satisfies the equation.

76. Solving an Equation. To find the value or values of the unknown quantity that will satisfy the equation is to *solve* the equation.

Thus if $x + 3 = 5$, then $x = 2$; for if we put 2 for x, the two members of the equation become $2 + 3$, and 5, which are the same in value. In other words, the equation then becomes an identity.

77. Root. A value of the unknown quantity that satisfies an equation is called a *root* of the equation.

Thus in $x - 4 = 9$, 13 is the root of the equation.

Simple equations have only one root. We shall later study equations that have more than one root, as we have already done in Book I. For example, the equation $x^2 - 3x + 2 = 0$ is satisfied if $x = 1$, and also if $x = 2$.

It would seem, from the work already covered in Book I, that the word "root" has two meanings. That is, we speak of the root of an equation and also of the square root and other roots of numbers. This difference is, however, only apparent, for when we find the positive square root of 144, we really find one of the roots of the equation $x^2 = 144$.

78. Axioms. A general statement admitted to be true without proof is called an *axiom.*

The axioms that suffice for most of the work in algebra are six in number.

1. *If equals are added to equals, the results are equal.*
2. *If equals are subtracted from equals, the results are equal.*
3. *If equals are multiplied by equals, the results are equal.*
4. *If equals are divided by equals, the results are equal.*
5. *Like powers or like roots of equals are equal.*
6. *Quantities equal to the same quantity are equal to each other.*

79. Typical Solution of a Problem. The sum of two numbers is 39, and one of them is three more than five times the other. Find the numbers.

Let x = the smaller number.

Then $5x + 3$ = the larger number.

Then $(5x + 3) + x = 39$.

Combining, $6x + 3 = 39$.

Subtracting 3, $6x = 36$.

Dividing by 6, $x = 6$.

Then $5x + 3 = 33$.

Therefore the required numbers are 6 and 33.

Check. Substituting in the statement of the problem,

$$6 + 33 = 39,$$

and

$$33 = 5 \times 6 + 3.$$

80. How to solve a Problem. To solve a problem,

(1) *Write a letter for the number sought;*

(2) *Use this letter in the statement of the problem;*

(3) *This will give an equation representing the problem;*

(4) *Find the value of the letter; that is, solve the equation.*

Exercise 49. Problems

1. If 1 more than a certain number is multiplied by the number, the product equals 3 more than the square of the number. What is the number?

2. If 2 less than a certain number is multiplied by the number, the product equals 4 less than the square of the number. What is the number?

3. If a certain number increased by 1 is multiplied by the number increased by 2, the product equals the sum of 8 and the square of the number. What is the number?

4. If a certain number increased by 2 is multiplied by the number increased by 3, the product equals 16 more than the square of the number. What is the number?

5. Twelve more than three times a certain number equals 45. Find the number.

6. Eight times a certain number equals 27 diminished by the number. Find the number.

7. Nine times a certain number equals 75 diminished by six times the number. Find the number.

8. Twelve times a certain number equals 80 diminished by eight times the number. Find the number.

9. Fifteen times a certain number equals 58 diminished by fourteen times the number. Find the number.

10. Nine times a certain number is diminished by 3.42, the result being 77.58. Find the number.

11. Ten times a certain number is diminished by six, the result being 36 more than four times the number. Find the number.

12. Twelve times a certain number is diminished by nine, the result being 54 more than nine times the number. Find the number.

13. Six times a certain number is increased by 15, the result being 45 more than four times the number. Find the number.

14. Seventeen times a certain number is increased by 13, the result being 77 more than nine times the number. Find the number.

15. Twenty-three times a certain number is increased by five times the number, the result being 160 more than eight times the number. Find the number.

16. From $5\frac{3}{4}$ times a certain number 7 is subtracted, the result being 93 more than $2\frac{1}{3}$ times the number. Find the number.

17. From $7\frac{3}{8}$ times a certain number $9\frac{1}{5}$ is subtracted, the result being $28\frac{13}{24}$ more than $5\frac{1}{4}$ times the number. Find the number.

81. Typical Solutions of Problems.

1. How much water must be added to 12 qt. of a 25% solution of ammonia to reduce it to a 10% solution?

Let x = the number of quarts added.

Then $12 + x$ = the total number of quarts.

Then $10\%(12 + x)$ = the number of quarts of ammonia.

But 25% of 12 = the number of quarts of ammonia.

Therefore $10\%(12 + x) = 25\%$ of 12.

Combining, $1.2 + 10\%x = 3.$

Subtracting 1.2, $10\%x = 1.8.$

Dividing by 10%, $x = 18.$

Therefore 18 qt. of water must be added.

Check. 25% of 12 qt. = 3 qt., the amount of ammonia.

12 qt. + 18 qt. = 30 qt., the amount of the mixture.

Furthermore, 3 qt. is 10% of 30 qt., as required.

2. A man starts from a certain place and travels on his bicycle at the rate of 16 mi. an hour. Forty-five minutes later an automobile starts after him at the rate of 24 mi. an hour. How long will it take the automobile to overtake him?

In 45 min. the bicycle has gone $\frac{3}{4}$ of 16 mi., or 12 mi.

The automobile goes 24 mi. an hour while the bicycle goes 16 mi., the automobile thus gaining on the bicycle 8 mi. an hour.

Let x = the number of hours required.

Since the automobile gains 8 mi. an hour,

$8x$ = the number of miles gained in x hr.

But 12 = the number of miles to be gained.

Therefore $8x = 12.$

Dividing by 8, $x = 1\frac{1}{2}.$

Therefore the automobile will overtake the bicycle in $1\frac{1}{2}$ hr.

Check. In $1\frac{1}{2}$ hr. the automobile will travel $1\frac{1}{2} \times 24$ mi., or 36 mi., and the bicycle will travel $1\frac{1}{2} \times 16$ mi., or 24 mi. Adding 12 mi., the start of the bicycle, 24 mi. + 12 mi. = 36 mi.

In all problems relating to motion the average rate is to be understood unless the contrary is stated. In other words, the motion referred to in such problems is to be taken as uniform.

Exercise 50. Problems

1. How much water must be added to a gallon of a 20% solution of ammonia to reduce it to a 10% solution?

2. How much water must be added to 32 gal. of alcohol, 95% pure, to reduce it to a 75% solution?

3. How many pounds of water must be added to 160 lb. of a 5% solution of salt to reduce it to a 4% solution? How many gallons, allowing 8.35 lb. of water to a gallon?

4. How much water must be added to a pint of a 5% solution of a certain medicine to reduce it to a 1% solution?

5. How much water must be added to a barrel of vinegar, 85% pure, to reduce it to a 50% solution?

6. How much ammonia must be added to a gallon of a solution that contains 20% pure ammonia so that the mixture shall contain 30% pure ammonia?

7. In an alloy of 90 oz. of silver and copper there are 6 oz. of silver. How much copper must be added so that 50 oz. of the new alloy may contain 2 oz. of silver?

8. In a mixture of 3 oz. of water and listerine there is 1 oz. of listerine. How much listerine must be added to make the new solution contain 75% pure listerine?

9. Our silver coins are 90% pure silver. How much pure silver must be melted with 100 oz. of silver alloy containing 80% pure silver to bring it to the standard required for our coinage?

10. In Ex. 9, how much pure silver must be melted with 200 oz. of silver alloy containing 75% pure silver to bring it to the 90% standard?

11. Air consists of five parts of nitrogen and one part of oxygen. How many cubic feet of oxygen are there in a schoolroom 40 ft. by 30 ft. by 10 ft.? How high would a room 40 ft. by 30 ft. be if it contained 2400 cu. ft. of oxygen?

12. Two men start from places 30 mi. apart and travel toward one another, the first at the rate of 7 mi. an hour and the second at the rate of 3 mi. an hour. How soon will they meet?

13. In Ex. 12, suppose the two men to travel at the same rate and to meet in 5 hr. Find the rate.

14. Two men start from places 600 mi. apart and travel toward one another, the first traveling 5 mi. an hour faster than the second. They meet in 8 hr. Find the rate of each.

15. A man starts on foot and walks at the rate of 3 mi. an hour. An hour after he has started a man sets out on horseback to overtake him and travels 6 mi. an hour. How soon will the second man overtake the first?

16. In Ex. 15, suppose the first man walks 4 mi. an hour and the second rides 6 mi. an hour. How many miles will the second man have to ride to overtake the first?

17. A train leaves St. Louis for Chicago at 8 A.M. and travels 35 mi. an hour. After an hour and a half a second train follows it on a parallel track at the rate of 47 mi. an hour. At what distance from St. Louis will the second train pass the first?

18. The distance from Albany to New York is 143 mi. A passenger train starts from New York at the same time that a freight train starts from Albany, and they travel toward one another. The passenger train makes 45 mi. an hour and the freight train 26½ mi. How soon will they meet and at what distance from New York?

19. In Ex. 18, suppose both trains to be going from Albany to New York, the freight having a start of 2 hr. How soon will the passenger train overtake the freight and at what distance from Albany?

20. In Ex. 18, suppose the trains travel 50 mi. and 25 mi. an hour, respectively, but the freight train is delayed 35 min. during the first hour. How soon will they meet?

21. An ocean liner making 21 knots an hour leaves Boston when a freight steamer making 8 knots an hour, and bound for the same port, is already 975 knots out. How long will it take the liner to overtake the freight steamer?

22. A river flows at the rate of 3 mi. an hour. A man who rows at the rate of 5 mi. an hour in still water wishes to row 1 mi. up the stream. How long will it take him?

23. A river flows at the rate of 2 mi. an hour. A man who rows at the rate of 5 mi. an hour in still water wishes to row 14 mi. down the stream. How long will it take him?

24. A river flows at the rate of 3 mi. an hour. A man rows a certain distance up the river in 12 hours and rows back to the starting place in 3 hours. If he keeps a uniform rate in rowing, what is this rate in still water?

25. A man finds that it takes his naphtha launch two hours to go 24 mi. with the tide, and four hours to go 8 mi. against the tide. Find the rate of the tide in miles per hour.

26. A certain kind of solder is composed of $1\frac{1}{8}$ oz. of tin to 1 oz. of lead. How many pounds of each are required to make 46 lb. of solder?

27. The circumferences of the front and rear wheels of a wagon are respectively 2 yd. and 3 yd. How far must the wagon go for the front wheel to make one revolution more than the rear wheel? to make 50 revolutions more than the rear wheel? to make $2\frac{1}{2}$ revolutions more than the rear wheel?

28. If an emery wheel will stand a surface speed of 2200 ft. per minute, how many revolutions can it make per minute if its diameter is 21 in.? (Use $\pi = 3\frac{1}{7}$.)

29. A boy in a manual-training school is making a bookcase. The distance from the top board to the bottom is 3 ft. $10\frac{1}{2}$ in., inside measure. He wishes to put in three shelves, each $\frac{7}{8}$ in. thick, so that the four book spaces will diminish successively by 1 in. from the bottom to the top. Find the spaces.

82. Clearing of Fractions. Multiplying both members of an equation by such a quantity as shall leave no fractions in the equation is called *clearing the equation of fractions.*

If $\frac{1}{2}x = 5,$

by multiplying by 2, $x = 10,$ an equation cleared of fractions.

If $\frac{x}{a} + \frac{x}{b} = ab,$

by multiplying by ab, $bx + ax = a^2b^2,$ an equation cleared of fractions.

To clear an equation of fractions, multiply both members of the equation by the lowest common multiple of the denominators.

If a fraction is preceded by a minus sign, change the sign of every term in the numerator when the denominator is removed.

After clearing of fractions, solve the equation in the usual manner.

Exercise 51. Problems

1. The sum of one half and three fourths of a certain number is two more than the number. Find the number.

2. There are two consecutive numbers, x and $x + 1$, such that half of the first plus one third of the second equals 7. Find the numbers.

3. There are three consecutive numbers, such that one seventh of the first, plus one fifth of the second, plus one eighth of the third equals half of the first. Find the numbers.

4. If the excess of a certain number over nine is divided by the sum of the number and nine, the quotient is $\frac{1}{4}$. Find the number.

5. One fifth of the excess of a certain number over three equals one sixth of the sum of the number and three. Find the number.

6. One half of a certain number, plus one fourth of the number, plus one eighth of the number, plus one sixteenth of the number equals 120. Find the number.

Solve the following equations:

7. $\dfrac{6x+7}{12} = \dfrac{3x-4}{4x-3} + \dfrac{x}{2}$.

8. $\dfrac{1}{x-3} + \dfrac{3}{x-9} + \dfrac{4}{6-x} = 0$.

9. $\dfrac{x-2}{3} + \dfrac{4-x}{5} + \dfrac{6-x}{7} = 0$.

10. One fifth of $6x+7$, divided by x, equals the quotient of $6x$ divided by $5x-6$. Find the value of x.

11. One twentieth of $4x+1$, diminished by the quotient of $2x-4$ divided by $7x-16$, equals one tenth of $2x-1$. Find the value of x.

12. One sixteenth of $11x-13$, diminished by one thirty-second of $22x-75$, equals one half of the quotient of $42x+7$ divided by $9x+7$. Find the value of x.

13. One thirty-sixth of $9x+20$ equals one fourth of x added to the quotient of $8x-11$ divided by $5x-1$. Find the value of x.

14. If from twice a certain number we subtract 4, and divide this difference by 5 more than the number, the quotient is $\frac{1}{3}$. Find the number.

15. If we divide 6 more than a certain number by 8 less than the number, the quotient is 15. Find the number.

16. The sum of 5 and a certain number is divided by the sum of 4 and the number. The quotient is $1\frac{1}{3}$. Find the number.

17. The sum of 7 and twice a certain number is divided by 6 less twice the number. The quotient is 1.5. Find the number.

18. If from one eighth of the sum of four times a certain number and three there is subtracted one fourth of twice the number less one, the result is the quotient of the excess of twice the number over one divided by the excess of five times the number over seven. Find the number.

83. Equations involving Decimal Fractions. In practical measurements we are coming more and more to use decimal fractions, and hence we meet these fractions in equations. They have the advantage that frequently we do not need to clear the equation of fractions.

1. Solve $0.7x - 0.3 = 0.05x + 1$.

Subtracting $0.05x$, $0.65x - 0.3 = 1$.

Adding 0.3, $0.65x = 1.3$.

Dividing by 0.65, $x = 2$.

Check. $0.7 \times 2 - 0.3 = 0.05 \times 2 + 1$,

for $1.4 - 0.3 = 0.1 + 1 = 1.1$.

2. Solve $0.4x + 0.3 - \dfrac{0.4x - 1}{x - 0.2} = \dfrac{0.2x - 0.1}{0.5}$.

Multiplying both terms of the two common fractions by 10,

$$0.4x + 0.3 - \frac{4x - 10}{10x - 2} = \frac{2x - 1}{5}.$$

The second member, $\dfrac{2x - 1}{5}$, evidently equals $0.4x - 0.2$, and hence

$$0.4x + 0.3 - \frac{2(2x - 5)}{2(5x - 1)} = 0.4x - 0.2.$$

Subtracting, $0.5 = \dfrac{2x - 5}{5x - 1}$.

Multiplying by $5x - 1$, $2.5x - 0.5 = 2x - 5$.

Subtracting, $0.5x = -4.5$.

Dividing by 0.5, $x = -9$.

Check. $-3.6 + 0.3 - \dfrac{-3.6 - 1}{-9 - 0.2} = \dfrac{-1.8 - 0.1}{0.5}$, for each member reduces to -3.8.

3. Solve $0.7x + 6.5 = 8 + 0.05x$.

If desired, the equation may be cleared of decimal fractions before transposing terms.

Multiplying by 100, $70x + 650 = 800 + 5x$.

Subtracting, $65x = 150$.

Dividing by 65, $x = 2\frac{4}{13}$.

Exercise 52. Equations involving Decimal Fractions

Examples 1 to 3, oral — Examples 4 to 23, written

1. Solve $x = 5 + 0.5x$; $x = 7 + 0.5x$; $x = 9 + 0.5x$.
2. Solve $x = 3 + 0.25x$; $x = 6 + 0.25x$; $x = 9 + 0.25x$.
3. Solve $0.2x = 1 - 0.3x$; $0.3x = 1 - 0.2x$; $0.4x = 1 - 0.1x$.

Solve the following equations and check:

4. $0.4x + 5 = 9$.
5. $0.7x + 3 = 11$.
6. $0.9x + 7 = 25$.
7. $0.09x + 7 = 25$.
8. $0.09x + 11 = 10.8$.
9. $0.5x - 6 = 0.2x$.
10. $0.7x - 5 = 0.2x$.
11. $2.5x - 24 = 1.3x$.
12. $3.9x - 4.8 = 2.7x$.
13. $5.3x - 0.68 = 1.9x$.

14. Solve the equation $0.3x + 7.2 = 4.8 - 0.02x$, (1) by clearing of decimal fractions before subtracting any terms from both members; (2) by first subtracting certain terms from both members and then dividing by the decimal coefficient of x.

Solve the following equations and check:

15. $0.7x - 3 + \frac{2}{5}x^2 = 0.3 + 0.4x(1 + x)$.
16. $3.48x - 8 + 0.7x = 1.1x + 5(8 + 0.01x)$.
17. $2.725x + 5(0.1x - 18) = 5(1.635 + 0.1x) - 90$.
18. $5x - 12.42748 = 4(-0.00517 - 0.3x) - 0.0034x$.
19. $2x - 4.03 = 2(2.0518 - 0.0001x) - 0.711x$.
20. $10x + 2 - 3(5x + \frac{1}{3}) = 4x + 0.6 - (x + 0.9)$.

21. Seven increased by 0.75 of a number equals 9.7. What is the number?

22. Four increased by 0.45 of a certain number equals 9.4. What is the number?

23. If to 0.72 of a certain number I add 0.4, the sum is 22. What is the number?

Exercise 53. Problems

1. A can do a piece of work in 4 da. and B can do it in 5 da. How many days will it take both together to do the work?

Let x = the number of days it will take both together.

Then $\frac{1}{x}$ = the part that both together can do in 1 da.,

$\frac{1}{4}$ = the part that A alone can do in 1 da.,

$\frac{1}{5}$ = the part that B alone can do in 1 da.,

and $\frac{1}{4} + \frac{1}{5}$ = the part that both together can do in 1 da.

Therefore $\frac{1}{4} + \frac{1}{5} = \frac{1}{x}$.

Solving, $x = 2\frac{2}{9}$.

2. A can do a piece of work in a days and B can do it in b days. How many days will it take both together to do the work? Evaluate the result for $a = 4$, $b = 5$.

3. A can do a piece of work in 4 da., B in 5 da., and C in 6 da. How many days will it take them to do it working together?

4. A can do a piece of work in a days, B in b days, and C in c days. How many days will it take them to do it working together? Evaluate the result for $a = 4$, $b = 5$, $c = 6$.

5. A and B together can build a wall in 12 da., A and C in 15 da., B and C in 20 da. In how many days can they build the wall if they all work together?

By working 2 da. each they build $\frac{1}{12} + \frac{1}{15} + \frac{1}{20}$ of it. Why?
Hence in 1 da. they build $\frac{1}{2}(\frac{1}{12} + \frac{1}{15} + \frac{1}{20})$ of it.

6. A and B together can build a wall in 10 da., A and C in 12 da., and B and C in 15 da. In how many days can they build the wall if they all work together?

7. A and B together can build a wall in d days, A and C in e days, B and C in f days. In how many days can they build the wall if they all work together? Evaluate the result for $d = 12$, $e = 15$, $f = 20$; for $d = 10$, $e = 12$, and $f = 15$.

8. A tank can be filled by two pipes in 2 hr. and 3 hr., respectively. In how many hours can it be filled if both pipes are open together?

Let x = the number of hours needed to fill the tank if both pipes are open together.

Then $\frac{1}{x}$ = the part filled in 1 hr. if both pipes are open together.

But $\frac{1}{2} + \frac{1}{3}$ = the part filled in 1 hr. if both pipes are open together.

Therefore $\frac{1}{2} + \frac{1}{3} = \frac{1}{x}$.

Solving, $x = 1\frac{1}{5}$.

$1\frac{1}{5}$ hr. = 1 hr. 12 min.

9. A tank can be filled by two pipes in 2 hr. and 3 hr., respectively, and emptied by a waste pipe in 4 hr. In how many hours can it be filled if all three pipes are open together?

Let x = the number of hours if all three pipes are open.

Then $\frac{1}{x}$ = the part filled in 1 hr. if all three pipes are open.

But $\frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ = the part that all can fill in 1 hr.

10. A tank can be filled by three pipes in a, b, and c hours, respectively. In how many hours can it be filled if all three pipes are open together?

11. Evaluate the result in Ex. 10 for $a = 8$, $b = 12$, $c = 16$; for $a = 1\frac{2}{3}$, $b = 3\frac{1}{3}$, $c = 5$; for $a = 2\frac{1}{4}$, $b = 3\frac{1}{2}$, $c = 4\frac{2}{3}$.

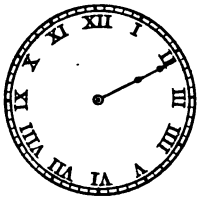
12. A tank can be filled by two pipes in a hours and b hours, respectively, and emptied by a waste pipe in c hours. In how many hours can it be filled if all three pipes are open together? Evaluate the result for $a = 2$, $b = 3$, $c = 4$.

13. Evaluate the first result in Ex. 12 for $a = 4$, $b = 5$, $c = 6$; for $a = 10$, $b = 12$, $c = 15$.

14. A cistern has three pipes. The first will fill the cistern in 6 hr., the second in 10 hr., and all three together in 3 hr. How long will it take the third pipe alone to fill it?

15. Solve Ex. 14, using a, b, and c instead of 6, 10, and 3.

16. Find the time between 2 and 3 o'clock when the hands of a clock are together.



Let x = the number of minute spaces passed over by the hour hand.

Then $12x$ = the number passed over by the minute hand.

Then $12x = x + 10$, since the hour hand is 10 spaces ahead at 2 o'clock.

$\therefore x = 1\frac{0}{11}$, and $12x = 10\frac{10}{11}$.

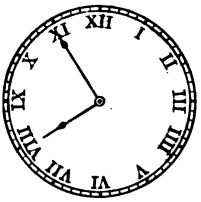
Therefore they will be together at $10\frac{10}{11}$ min. after 2.

Find the time when the hands of a clock are together between:

17. 3 and 4. **19.** 5 and 6. **21.** 9 and 10.

18. 4 and 5. **20.** 6 and 7. **22.** 11 and 12.

23. Find the time between 7 and 8 o'clock when the hands of the clock are at right angles to each other.



Let x = the number of minute spaces passed over by the hour hand.

Then $12x$ = the number passed over by the minute hand.

But the minute hand must gain 35 spaces to catch up with the hour hand after 7 o'clock, or $35 + 15$ spaces to be 15 spaces (a right angle) ahead, or $35 - 15$ spaces to be 15 spaces behind it.

Then $12x = x + 50$, when the minute hand is 15 spaces ahead,

or $12x = x + 20$, when it is 15 spaces behind.

$\therefore x = 5\frac{5}{11}$ or $2\frac{2}{11}$,

and $12x = 54\frac{6}{11}$, or $21\frac{9}{11}$.

But $54\frac{6}{11}$ min. = 54 min. $32\frac{8}{11}$ sec., and $21\frac{9}{11}$ min. = 21 min. $49\frac{1}{11}$ sec.

Therefore the time is 7 hr. 54 min. $32\frac{8}{11}$ sec.

or 7 hr. 21 min. $49\frac{1}{11}$ sec.

Find the time when they are at right angles between:

24. 6 and 7. **25.** 2 and 3. **26.** 11 and 12.

Find the time when they are opposite between:

27. 4 and 5. **28.** 10 and 11. **29.** 9 and 10.

30. Find two consecutive numbers, x and $x+1$, such that $\frac{1}{2}$ of the smaller exceeds $\frac{1}{3}$ of the larger by 5.

31. Find three consecutive numbers such that their sum is 99; such that their sum is 159.

32. Find three consecutive numbers such that the sum of the largest and smallest is 20 more than the other number.

33. Find four consecutive numbers such that their sum is 126; such that their sum is 210.

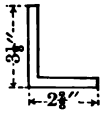
34. Find four consecutive numbers such that one sixth of the smallest is two more than one eleventh of the largest.

35. The difference of two numbers is 30, and if the greater is divided by the less, the quotient is 11. Find the numbers.

36. The difference of two numbers is 12, and if the greater is divided by the less, the quotient is 2 and the remainder 5. Find the numbers.

37. A man has 6 hr. at his disposal. How far may he ride in a trolley car at 9 mi. an hour so as to return in time, walking back at the rate of 3 mi. an hour?

38. The number of square inches of area of a cross section of an L beam is 3 minus the square of the thickness. Find the thickness.



39. If to x sevenths I add $\frac{5}{8}$, the result is $1\frac{3}{56}$. Find the value of x.

40. If to seven $x$ths I add $\frac{5}{8}$, the result is $1\frac{1}{2}$. Find the value of x.

41. If I divide 8 by $x+1$, the result is the same as that of dividing 4 by $x-1$. Find the value of x.

42. The moon as seen from the earth appears to complete the circuit of the heavens in 27 da. 7 hr. 43 min. 4.68 sec., and the sun in 365 da. 5 hr. 48 min. 47.8 sec., in the same direction. Assuming their apparent motions to be uniform, find the time from one full moon to the next.

CHAPTER VI

SIMULTANEOUS SIMPLE EQUATIONS

84. Indeterminate Equations. If we have one simple equation containing one unknown quantity, we can solve by the methods already studied. If it contains two unknown quantities, we can find one quantity in terms of the other.

For example, if $2x - 3y = 8$, we can see that $x = \frac{8 + 3y}{2}$, and that $y = \frac{2x - 8}{3}$; but this does not tell us the value of either.

Indeed, if $y = 1$ we see that $x = 5\frac{1}{2}$; if $y = 2$, $x = 7$; if $y = 4$, $x = 10$, and so on, there being an indefinite number of values of x and y that satisfy this equation.

An equation that has an indefinite number of roots is called an *indeterminate equation.*

An equation that is not indeterminate is said to be *determinate.*

85. Two Simple Equations. If we have two simple equations containing two unknown quantities, we can usually derive from them a single equation containing only one unknown quantity.

For example, suppose $x + y = 17$,

and $x - y = 9$.

Adding member for member, $2x = 26$;

whence $x = 13$.

And because $x + y = 17$,

we have $13 + y = 17$;

whence $y = 4$.

86. Simultaneous Equations. Two or more equations that have a set of values for the unknown quantities in common are called *simultaneous equations.*

87. Elimination. The process by which we cause an unknown quantity to disappear in deriving an equation from a system of equations is called *elimination.*

1. Solve the system of equations

$$2x + 3y = 27 \quad (1)$$

$$5x - 2y = 1 \quad (2)$$

Multiplying (1) by 2, $4x + 6y = 54$

Multiplying (2) by 3, $15x - 6y = 3$

Adding, $19x = 57$

Dividing by 19, $x = 3.$

Substituting 3 for x in (1), $6 + 3y = 27.$

Solving for y, $y = 7.$

In this solution y is eliminated by *addition.*

2. Solve the system of equations

$$3x + 2y = 23 \quad (1)$$

$$2x + 3y = 27 \quad (2)$$

Multiplying (1) by 3, $9x + 6y = 69$ (3)

Multiplying (2) by 2, $4x + 6y = 54$ (4)

Subtracting (4) from (3), $5x = 15$

Dividing by 5, $x = 3.$

Substituting in (1) or (2), $y = 7.$

In this solution y is eliminated by *subtraction.*

3. Solve the system of equations

$$5x + 2y = 34 \quad (1)$$

$$7x - 3y = 7 \quad (2)$$

From (1), $$y = \frac{34 - 5x}{2}. \quad (3)$$

Substituting in (2), $$7x - 3 \cdot \frac{34 - 5x}{2} = 7.$$

Solving for x, $x = 4.$

Substituting in (3), $$y = \frac{34 - 5 \cdot 4}{2}$$

$$= 17 - 5 \cdot 2 = 7.$$

In this solution y is eliminated by *substitution.*

4. Solve the system of equations

$$\frac{3}{x}+\frac{4}{y}=1 \qquad (1)$$

$$\frac{21}{x}-\frac{2}{y}=2 \qquad (2)$$

Multiplying (2) by 2, $\frac{42}{x}-\frac{4}{y}=4.$

Adding (1), $\frac{45}{x}=5.$

Multiplying by x and dividing by 5,

$$9=x.$$

Substituting 9 for x in (1), $\frac{3}{9}+\frac{4}{y}=1,$

or $\frac{4}{y}=\frac{2}{3}.$

Solving for y, $y=6.$

88. Directions for Elimination. From the preceding solutions we see that we may proceed as follows:

Multiply both members of the equations by such numbers as will make the coefficients of one of the unknown quantities numerically equal.

If these coefficients have opposite signs, add the equations member for member; if they have the same signs, subtract.

Solve the resulting equation.

Substitute the result thus found in the simpler of the two given equations and solve for the other unknown quantity.

Or find the value of one unknown quantity in terms of the other in one equation, and substitute it in the other equation, and then solve this equation. From the result the other root can then be found.

Check the results by substituting in both of the given equations.

We commonly say "Multiply the equation," meaning thereby "Multiply both members of the equation." Similarly, we speak of adding equations and subtracting equations.

Exercise 54. Simultaneous Equations

Solve the following equations:

1. $7x + 3y = 23$
   $5x + 3y = 19$

2. $10x + 3y = 75$
   $8x - y = 43$

3. $11x + 21y = 288$
   $16x - 30y = -126$

4. $1.5P + 1.25Q = 19$
   $1.2P + 0.75Q = 13.2$

5. $0.5M - 0.3N = 2$
   $0.3M - 0.4N = 0.1$

6. $x + 3\frac{1}{2}y = 17$
   $3\frac{1}{2}x + y = 37$

Solve the following equations without first clearing of fractions:

7. $\frac{3}{x} + \frac{4}{y} = 3$
   $\frac{9}{x} - \frac{8}{y} = -1$

8. $\frac{15}{x} + \frac{6}{y} = 8$
   $\frac{21}{x} - \frac{12}{y} = 1$

9. $\frac{x}{3} + \frac{y}{4} = 6$
   $\frac{x}{6} - \frac{y}{3} = -2\frac{1}{2}$

10. $\frac{15}{2x} + \frac{1}{y} = 2$
    $\frac{25}{2x} - \frac{3}{y} = 1$

11. $\frac{43.2}{5x} - \frac{6.2}{5y} = \frac{4}{5}$
    $\frac{64.8}{5x} + \frac{12.4}{5y} = 2.6$

12. $\frac{1}{2x} + \frac{1}{y} = p$
    $\frac{1}{2x} - \frac{1}{y} = q$

13. $\frac{a}{x} + \frac{b}{y} = c$
    $\frac{b}{x} + \frac{a}{y} = d$

14. $\frac{1}{ax} + \frac{1}{by} = c$
    $\frac{1}{bx} - \frac{1}{ay} = d$

15. $\frac{a}{bx} + \frac{b}{ay} = a + b$
    $\frac{b}{x} + \frac{a}{y} = a^2 + b^2$

16. $\frac{a}{x} + \frac{b}{y} = c$
    $\frac{a'}{x} + \frac{b'}{y} = c'$

Solve the following equations:

17. $\dfrac{5x-2}{15-4y}=\dfrac{2.5x+3}{9-2y}$

$\dfrac{2x+3y+1}{4x+6y-8}=\dfrac{3x-7y+8}{6x-14y+6}$

18. $\dfrac{6x-7}{19-10y}=\dfrac{3x-4}{11-5y}$

$\dfrac{6x-10y-17}{3x-5y+2}=\dfrac{4x-14y-5}{2x-7y+12}$

19. $\dfrac{5(x-1.2y)}{13}+2=4y-3x$

$\dfrac{5(x-1.2y)}{6}+12=2y-\dfrac{2y-3x}{4}$

20. $8y-\dfrac{4(4+15y)}{3x-1}=\dfrac{8xy-53.5}{x+2.5}$

$2+3(3y+2x)=\dfrac{3(3y+2x)(3y-2x)+38}{1-2x+3y}$

21. $\dfrac{x+y}{3}+\dfrac{x-y}{3}=2.0944$

$\dfrac{3x+4y-1}{4}+\dfrac{2x-7y-3}{3}=2.7762$

22. $\dfrac{2.471x+4.11y-3}{4}-\dfrac{5.004x-3.012y}{3}=3.244$

$\dfrac{4.005x-3.034y+9}{4}+\dfrac{11.07x+1.233y-9}{3}=7.59$

23. $\dfrac{3.072x-4.002y+0.929}{2}+\dfrac{4.003x+8.066y+8}{4}=5.009$

$\dfrac{1.781x+3.024y+0.707}{4}+\dfrac{2.755x+2.208y+1.141}{5}=2$

24. Show that the system of equations $\dfrac{x+4}{x-2}=\dfrac{y+1}{y-1}$ and $\dfrac{3x-12}{3y-5}=\dfrac{2x+5}{2y+1}$ has an indefinite number of roots.

89. Determinant. If we solve the two simultaneous equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

we obtain $x = \dfrac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}$, $y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$.

The denominator $a_1b_2 - a_2b_1$ is also written in this form: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This is called a *determinant.*

Similarly, $c_1b_2 - c_2b_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

90. Diagonals. The diagonal a_1, b_2, is called the *principal diagonal* of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, and a_2, b_1, is called the *secondary diagonal.*

91. Elements. The several symbols, a_1, a_2, b_1, b_2, $\cdots$, that enter into a determinant are called its *elements.*

The *expansion* of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is $a_1b_2 - a_2b_1$. The terms a_1b_2, a_2b_1, of this expansion are called the *terms of the determinant.* The *order* of a determinant is the number of elements in a row (horizontal) or column (vertical).

92. Expansion of a Determinant of the Second Order. *A determinant of the second order equals the product of the terms in the principal diagonal minus the product of the terms in the secondary diagonal.*

93. Equations solved by Determinants. From § 89 we see that we can write out the roots of two simultaneous equations at sight. That is, given

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

we know at once that

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

94. Illustrative Problem. Solve the equations

$$2x - 3y = 4$$
$$3x + 2y = 32$$

We have, by § 93,

$$x = \frac{\begin{vmatrix} 4 & -3 \\ 32 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}} = \frac{4 \cdot 2 - (-3) \cdot 32}{2 \cdot 2 - (-3) \cdot 3} = \frac{8 + 96}{4 + 9} = \frac{104}{13} = 8,$$

and

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & 32 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix}} = \frac{2 \cdot 32 - 3 \cdot 4}{2 \cdot 2 - (-3) \cdot 3} = \frac{64 - 12}{4 + 9} = \frac{52}{13} = 4.$$

It will be noticed that the denominator is formed in each case by taking the coefficients of x and y as they stand in the two equations. The numerator of the value for x is the same as the denominator, if the coefficients of x are replaced by the absolute term; and similarly for the numerator of the value of y.

Exercise 55. Solutions by Determinants

Solve the following by determinants:

1. $6x + 5y = 46$
   $10x + 3y = 66$

2. $2x + 7y = 52$
   $3x - 5y = 16$

3. $4x + 9y = 79$
   $7x - 17y = 40$

4. $2x - 7y = 8$
   $9x - 4y = -19$

5. $x + 4y = 16$
   $4x + y = 34$

6. $7x + 2y = -12$
   $3x + 12y = 0$

7. $5x - 2y = 78$
   $x - 3y = -104$

8. $4x + 3y = 60$
   $20x - 19y = 28$

9. $2x + 3y = 8$
   $0.3x + 0.7y = 0.7$

10. $x - 4\frac{1}{2}y = 5\frac{1}{2}$
    $3x - 4y = 7$

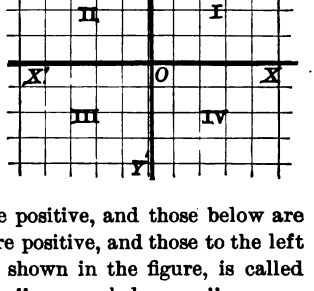
11. $9x + y = 98$
    $7x - 2y = 15$

12. $4x + 7y = 121$
    $8x - 3y = 55$

95. Terms used in Graphs of Equations. We found in Book I that we could represent an equation graphically.

The expressions "plotting an equation" "graphing an equation," and "drawing the graph of an equation" are all used, and all have the same meaning.

In the annexed figure, $X'X$ is called the *axis of* x, and YY' the *axis of* y. The point O is called the *origin.* The distance of a point above or below the axis of x is called its *ordinate,* and the distance to the right or left of the axis of y is called its *abscissa.* The ordinate and abscissa together are called the *coördinates* of the point.



The ordinate is the distance from P to OX; the abscissa is the distance from P to OY. Ordinates above $X'X$ are positive, and those below are negative. Abscissas to the right of YY' are positive, and those to the left are negative. Paper ruled in squares, as shown in the figure, is called "squared paper," "coördinate paper," or "cross-ruled paper."

96. Plotting an Equation. Plot the equation $3x + 4y = 7$.

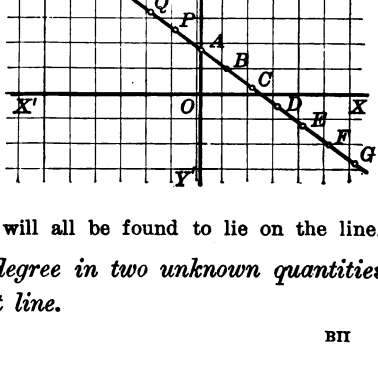
Solving for y we have

$$y = \tfrac{1}{4}(7 - 3x).$$

That is, y is a function of x (§ 64), and will vary as x varies.

If $x = 1$, $y = \frac{1}{4}(7 - 3 \times 1) = 1$; if $x = 0$, $y = \frac{1}{4}$ of $7 = 1\frac{3}{4}$.

We have, therefore, two points of the line $x = 1$ and $y = 1$, and $x = 0$ and $y = 1\frac{3}{4}$, indicated thus: $(1, 1)$ and $(0, 1\frac{3}{4})$. Drawing a straight line through these points we have the graph. Other points may be fixed, as shown in the figure, and they will all be found to lie on the line.

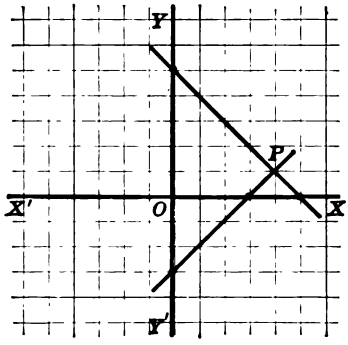


An equation of the first degree in two unknown quantities always represents a straight line.

97. Two Linear Equations. Two linear equations involving two variables are represented by two straight lines. These lines can intersect in only one point. Therefore, in general,

The graphs of two linear equations involving two variables have only one point in common;

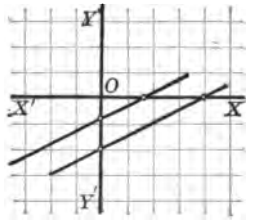
Two linear equations involving two variables have only one pair of values of the variables in common.



Thus the graphs of $x + y = 5$ and $x - y = 3$ intersect at P. The coördinates of P are 4 and 1. Hence $x = 4$, $y = 1$.

There are an infinite number of points on each graph, but there is only one point on both graphs. Similarly, there are an infinite number of values of x and y that will satisfy each equation, but there is only one value that will satisfy both equations.

98. Inconsistent Equations. If we plot the equations $x - 2y = 4$ and $3x - 6y = 5$, we shall have two parallel lines. Such lines have no point in common. Considering the equations, we see that the second one reduces to $x - 2y = 1\frac{2}{3}$. The equations are therefore *inconsistent*, since $x - 2y$ cannot equal both 4 and $1\frac{2}{3}$.



99. Equivalent Equations. If we plot the equations $2x + 4y = 5$ and $x + 2y = 2\frac{1}{2}$, we shall find that one graph coincides with the other. They have an infinite number of points in common, and therefore an infinite number of values of x and y satisfy the equations, every pair of roots of either being a pair of roots of the other. The equations are therefore *equivalent.*

Exercise 56. Graphs

Using squared paper, locate the following points:

1. 3, 7. **3.** 7, − 3. **5.** 2.5, 3.7. **7.** −2.2, − 3.5.
2. − 2, 5. **4.** − 4, − 6. **6.** 3.5, − 2. **8.** − 3.2, 5.2.

Plot the following equations:

9. $y = 2x + 3$. **11.** $y = -5x + 2$. **13.** $x + 2y = 5$.
10. $y = 3x - 2$. **12.** $y = -3x - 4$. **14.** $2x - 3y = 7$.

15. In the equation $y = 5$, x does not appear. Hence x may have any value, but y is always 5. This equation, therefore, represents a line parallel to what axis, and how far from it? Draw the graph.

Plot the following equations:

16. $y = 3$. **18.** $y = -4$. **20.** $x = 0$. **22.** $x = -3\frac{1}{2}$.
17. $y = 0$. **19.** $y = -2\frac{1}{2}$. **21.** $x = 2\frac{1}{2}$. **23.** $x = -0.5$.

Plot the following equations and find their roots by measuring the abscissas and ordinates of the points of intersection:

24. $x + y = 0$
$x - y = 14$

25. $2x + 4y = 0$
$5x - 3y = 13$

26. $7x - 3y = 15$
$4x + 5y = 22$

27. $3x + y = 19$
$2x - y = 1$

28. $5x + 9y = 17$
$7x - 2y = -20$

29. $3x + 4y = -9$
$5x - 2y = 24$

Plot the following equations and show that they are inconsistent:

30. $2x + 3y = 4$
$x + 1.5y = 3$

31. $7x - 4y = 5$
$1.75x - y = 2$

32. $5y = 8x - 3$
$7y = 11.2x - 6$

33. $9x = 4y + 1$
$2\frac{1}{4}x = y + 2$

Plot the following equations and show that they are equivalent:

34. $2x + 3y = 4$
$3x + 4\frac{1}{2}y = 6$

35. $5x - 2y = 7$
$7\frac{1}{2}x - 3y = 10\frac{1}{2}$

36. $8y = 2x - 3$
$12y = 3x - 4\frac{1}{2}$

37. $7x = 3y + 1$
$17\frac{1}{2}x = 7\frac{1}{2}y + 2\frac{1}{2}$

Plot the following equations and discuss their roots, giving their values wherever there are common roots:

38. $2x + 3y = 7$
$4x - 5y = 3$

39. $x - 2y = 4$
$2x - y = 5$

40. $7x + 2y = 30$
$-3x + y = 2$

41. $3x - 5y = 51$
$2x + 7y = 3$

42. $5x + 4y = 58$
$3x + 7y = 67$

43. $3x + 2y = 39$
$-2x + 3y = 13$

44. $3x - 4y = -5$
$4x - 5y = 1$

45. $11x + 3y = 100$
$4x - 7y = 4$

46. $x + 15y = 53$
$3x + y = 27$

47. $3x - 4y = 2$
$7x - 9y = 7$

48. $7x - 5y = 24$
$4x - 3y = 11$

49. $3x + 2y = 32$
$20x - 3y = 1$

50. $11x - 7y = 37$
$8x + 9y = 41$

51. $7x + 5y = 60$
$13x - 71y = 10$

52. $2x + \frac{1}{2}y = 21$
$3x - \frac{1}{7}y = 19$

53. $4x + 3y = 20$
$8x - 9y = -10$

54. $2x - y = 8$
$3x + 2y = 19.7$

55. $8x + 3y = 6$
$2x - 3y = -1$

56. $4x + 5y = 32$
$2x + 15y = 23\frac{1}{2}$

57. $2x - 7y = 33$
$3x - 5y = 11$

100. Three or more Simultaneous Equations. Three or more simultaneous equations may be solved by eliminating one unknown quantity from one pair of the equations, the same quantity from another pair, and so on until two equations involving two unknown quantities result, these being solved as usual.

Solve the equations

$$5x + 2y - 4z = -3 \quad (1)$$

$$3x - 3y + 5z = 12 \quad (2)$$

$$4x + 5y + 2z = 20 \quad (3)$$

We may eliminate z by combining (1) and (2), as follows:

Multiplying (1) by 5, $25x + 10y - 20z = -15.$ (4)

Multiplying (2) by 4, $12x - 12y + 20z = 48.$ (5)

Adding, $37x - 2y = 33.$ (6)

We may eliminate z between (1) and (3), as follows:

Multiplying (3) by 2, $8x + 10y + 4z = 40.$ (7)

Adding (1) and (7), $13x + 12y = 37.$ (8)

We now have two equations, (6) and (8), involving x and y. We may eliminate y as follows:

Multiplying (6) by 6, $222x - 12y = 198.$ (9)

Adding (8) and (9), $235x = 235.$

Therefore $x = 1.$

Substituting in (8), $13 + 12y = 37.$

Hence $12y = 24,$

and $y = 2.$

Substituting $x = 1$, $y = 2$, in (1), $5 + 4 - 4z = -3.$

Solving for z, $z = 3.$

Therefore $x = 1,\ y = 2,\ z = 3.$

Check. Substituting in (1), (2), and (3), we have

$$5 + 4 - 12 = -3,$$

$$3 - 6 + 15 = 12,$$

$$4 + 10 + 6 = 20.$$

In checking, it is usually sufficient to substitute in the equation which has been used least in the solution. To be sure of the results, however, it is necessary to substitute in all three of the original equations.

Exercise 57. Simultaneous Equations

Solve the following equations:

1. $x+y=2$
$y+z=-2$
$z+x=12$

2. $x+y=5$
$y+z=3$
$z+x=7$

3. $2y+z=9$
$z-2y=1$
$x+y+z=1$

4. $x+3=5-4y$
$x+z=3y$
$8y-4=z$

5. $x-\frac{b}{a}y+\frac{c}{a}z=a$
$x+\frac{b}{a}y-\frac{c}{a}z=\frac{b^2}{a}$
$-ax+by+cz=c^2$

6. $\frac{x}{b}+\frac{y}{c}+\frac{z}{a}=3$
$\frac{x}{b}+\frac{y}{c}-\frac{z}{a}=1$
$\frac{x}{b}-\frac{y}{c}+\frac{z}{a}=1$

7. $\frac{x}{6}+\frac{y}{3}+\frac{z}{4}=36$
$\frac{x}{9}+\frac{y}{15}+\frac{z}{20}=10$
$\frac{x}{4}+\frac{y}{2}+\frac{z}{10}=43$

8. $x+y+z=7.77$
$3x+y-z=6.51$
$7x-2y+3z=-11.49$

9. $3(z-1)=2(y-1)$
$4(x+y)=9z-4$
$2y-9=7(5x-3z)$

10. $0.7x-1.1y+0.2z=1$
$x-0.3y-0.5z=1.5$
$1.2x-0.1y-0.6z=3.1$

11. $(a+b)x+(a-b)z=2bc$
$(b+c)y+(a+b)x=2ac$
$(a-b)z+(b+c)y=2ab$

12. $x+\frac{b}{a}y+\frac{c}{a}z=1$
$x-\frac{b}{a}y-\frac{c}{a}z=\frac{b}{a}$
$ax+cy+bz=c$

13. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=12$
$\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=14$
$\frac{3}{x}+\frac{5}{y}-\frac{7}{z}=-6$

14. $\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=2.9$
$\frac{5}{x}-\frac{6}{y}-\frac{7}{z}=-10.4$
$\frac{9}{y}+\frac{10}{z}-\frac{8}{x}=14.9$

Solve the following equations:

15. $w+x+y+z=10$
$w+x+y-z=2$
$w+x-y+z=4$
$w-x+y+z=6$

16. $w+x+y+z=18$
$w-2x+y+2z=12$
$w+3x-2y-z=-1$
$4w-x+3y-5z=-7$

17. $w+2x+3y+4z=3$
$2w+3x+4y+5z=3$
$6w+7x-8y+9z=35$
$64w+16x+4y+z=-54$

18. $w+x+y=6$
$x+y+z=6$
$y+z+w=6$
$z+w+x=6$

19. $w+2x+3y=12$
$w+4x-3y=14$
$w+3x+y=14$
$x+y+z=9$

20. $u+v+x=7$
$2v+2x+2y=8$
$3x+3y+3u=24$
$4y+4u+4v=28$

21. If one of two numbers is decreased by 1.3, the result is the other number. If three times the larger number is decreased by the smaller number, the result is 4.7. Find the numbers.

Let x = the larger number,
and y = the smaller number.
Then $x-1.3=y,$ (1)
and $3x-y=4.7.$ (2)

Substituting the value of y from (1) in (2),

$$3x-x+1.3=4.7.$$

Therefore $2x=3.4,$
and $x=1.7.$
Therefore $y=x-1.3=1.7-1.3=0.4.$

Therefore the numbers are 1.7 and 0.4.

22. If to twice one number we add three times a second number, the result is 14.7. If from twice the first number we subtract the second number, the result is 2.3. Find the numbers.

23. If to four times one number we add twice a second number, the result is $13\frac{1}{2}$. If from five times the first number we subtract four times the second number, the result is $5\frac{1}{2}$. Find the numbers.

24. A certain fraction becomes equal to $\frac{2}{3}$ if 7 is added to the numerator, and equal to $\frac{3}{5}$ if 7 is subtracted from the denominator. Find the fraction.

25. If the larger of two numbers is divided by the smaller, the quotient is 3 and the remainder 3; but if the smaller is divided by the larger, the quotient is 0.266 and the remainder 0.01. Find the numbers.

Let $x =$ the larger number,

and $y =$ the smaller number.

Then $$\frac{x}{y} = 3 + \frac{3}{y}, \tag{1}$$

and $$\frac{y}{x} = 0.266 + \frac{0.01}{x}. \tag{2}$$

Clearing (1) of fractions, $x - 3 = 3y$.

Clearing (2) of fractions,

$$y - 0.01 = 0.266x.$$

Eliminating y, $0.202x = 3.03$.

Dividing by 0.202, $x = 15$.

Substituting in (1), $\frac{15 - 3}{y} = 3$.

Solving, $y = 4$.

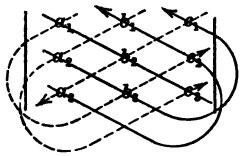
Therefore the numbers are 15 and 4.

26. If the larger of two numbers is divided by the smaller, the quotient is 3 and the remainder 3; but if the smaller is divided by the larger, the quotient is 0.2 and the remainder 2.6. Find the numbers.

27. If the smaller of two numbers is divided by the larger, the quotient is 0.8 and there is no remainder; but if the larger is divided by the smaller, the quotient is 1.2 and the remainder 0.36. Find the numbers.

28. If the larger of two numbers is divided by the smaller, the quotient is 4 and the remainder 9; but if twenty times the smaller is divided by twice the larger, the quotient is 2 and the remainder 152. Find the numbers.

101. Expansion of a Determinant of the Third Order. A determinant of the third order is expanded by taking the sum of the products of the three elements on the full lines, and subtracting from it the sum of the products of the three elements on the dotted lines in the diagram.



The expansion is, therefore,

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

102. Equations solved by Determinants. When we solve the equations

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

we find that

$$x = \frac{k_1b_2c_3 + k_2b_3c_1 + k_3b_1c_2 - k_1b_3c_2 - k_2b_1c_3 - k_3b_2c_1}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1},$$

where the denominator is the expansion given in § 101.

In other words,
$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

We see that, as in § 93, the denominator is formed by taking the coefficients of x, y, and z as they stand in the three equations. The numerator for the value of x is the same as the denominator if the coefficients of x are replaced by the absolute terms.

If we should solve for y and z, we should find that the values of these letters are obtained in the same manner; that is,

$$y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

103. Illustrative Problem. Solve the equations

$$5x + 2y - 4z = -3$$
$$3x - 3y + 5z = 12$$
$$4x + 5y + 2z = 20$$

By § 102 we have

$$x = \frac{\begin{vmatrix} -3 & 2 & -4 \\ 12 & -3 & 5 \\ 20 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 5 & 2 & -4 \\ 3 & -3 & 5 \\ 4 & 5 & 2 \end{vmatrix}} = \frac{-235}{-235} = 1,$$

$$y = \frac{\begin{vmatrix} 5 & -3 & -4 \\ 3 & 12 & 5 \\ 4 & 20 & 2 \end{vmatrix}}{-235} = \frac{-470}{-235} = 2,$$

$$z = \frac{\begin{vmatrix} 5 & 2 & -3 \\ 3 & -3 & 12 \\ 4 & 5 & 20 \end{vmatrix}}{-235} = \frac{-705}{-235} = 3.$$

The gain over the ordinary methods of eliminating is here very apparent.

Exercise 58. Simultaneous Equations

Expand the following determinants:

1. $\begin{vmatrix} 1 & 4 & 5 \\ 2 & 0 & 2 \\ 3 & 1 & 7 \end{vmatrix}$ **2.** $\begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 3 & 1 \end{vmatrix}$ **3.** $\begin{vmatrix} 5 & 0 & 0 \\ 2 & 7 & 8 \\ 9 & 2 & 6 \end{vmatrix}$ **4.** $\begin{vmatrix} 2 & -4 & 0 \\ 3 & 2 & 1 \\ 1 & -3 & 1 \end{vmatrix}$

Solve the following by determinants:

5. $3x - 4y + 2z = 1$
$2x + 3y - 3z = -1$
$5x - 5y + 4z = 7$

6. $x + y + z = 4$
$3x - 2y + 4z = 9$
$5x + 3y - 2z = 4$

7. $4x - 7y + z = 16$
$3x + y - 2z = 10$
$5x - 6y - 3z = 10$

8. $2x + y - z = 7$
$3x + 2y + 3z = 20$
$2x + 5y + 4z = 21$

9. In the equation $B = xA + y$, it is known that when $B = 18$ the value of A is 8, and when $B = 46$ the value of A is 16. Find the values of x and y. Check by substituting in both equations.

10. In the equation $F = aB + c$, it is known that when $F = 110$ the value of B is 8, and when $F = 210$ the value of B is 16. Find the values of a and c.

11. In the equation $R = aE + b$, it is known that when $R = 40$ the value of E is 10, and when $R = 220$ the value of E is 50. Find the values of a and b.

12. Of two squares of carpet, the perimeter of one is 44 ft. more than that of the other, and the sum of the perimeters is 100 ft. Find the side of each.

13. Two pictures are framed in the same manner. The first is 1 ft. 6 in. by 2 ft., and the second is 2 ft. by 2 ft. 6 in. The frame, glass, and labor for the first cost \$1.50, of which 36¢ was for labor and the extra molding for the corners; and that for the second cost \$2.10, of which 42¢ was for labor and the corners. What is the price of the glass per square foot, and the price of the frame per linear foot?

14. The dimensions of a rug are such that if the length were $1\frac{1}{2}$ ft. less, the rug would be a square; but if the width were $5\frac{1}{2}$ in. less, the length would be double the width. Find the dimensions.

15. A lady paid 76¢ for some sugar and tea. The sugar cost 6¢ a pound and the tea 80¢ a pound. The sugar weighed twelve times as much as the tea. How many pounds of each did she buy?

16. A lady bought 12 lb. of sugar, 2 lb. of tea, and 5 doz. eggs, paying \$3.74 for all. The tea cost ten times as much per pound as the sugar. If the tea had cost 10¢ less per pound, she could have bought 2 doz. eggs for the cost of 1 lb. of tea. Find the price of the sugar and tea per pound, and of the eggs per dozen.

17. The sum of the three digits of a number is 8. The digit in the units' place exceeds that in the tens' by 3, and if 396 is added to the number, the order of the digits will be reversed. Find the number.

Let h = the hundreds' digit,

t = the tens' digit,

and u = the units' digit.

Then $100h + 10t + u$ = the number.

But $h + t + u = 8,$ (1)

$u - t = 3,$ (2)

and $100h + 10t + u + 396 = 100u + 10t + h,$ (3)

since the order of the digits is now reversed.

From (1) and (2), $h + 2u = 11.$

From (3), $99h - 99u = -396,$

whence $h - u = -4.$

Eliminating h, $u = 5.$

From (2), $t = u - 3 = 5 - 3 = 2.$

From (1), $h = 8 - t - u = 8 - 5 - 2 = 1.$

Therefore the digits are 1, 2, 5, and the number is 125.

18. The sum of the three digits of a number is 12. The sum of the hundreds' and tens' digits is four more than the units' digit. The sum of the hundreds' and units' digits is four more than the tens' digit. Find the number.

19. The sum of the three digits of a number is 9. The sum of the hundreds' and units' digits is 6, and if 198 is added to the number, the order of the digits will be reversed. Find the number.

20. The sum of the three digits of a number is 6. The sum of the hundreds' and tens' digits is 3, and the units' digit is three more than the tens' digit. Find the number.

21. The sum of the three digits of a number is 18. The value of the number is not changed if the order of the digits is reversed, and the sum of the hundreds' and units' digits equals the sum of the units' and tens' digits. Find the number.

22. In any triangle the sum of the three angles is 180°. In a certain triangle ABC the sum of angles A and B is 130°, and the sum of angles B and C is 110°. Find the number of degrees in each of the three angles.

23. In Ex. 22, if angle A is 30° larger than angle B, and angle B is 30° larger than angle C, find the three angles.

24. In a certain triangle ABC angle A is 100° larger than angle B and 110° larger than angle C. Find the number of degrees in each of the three angles.

25. Of the three angles of a triangle, the sum of the second and third equals twice the first, and the difference between the second and first equals five times the third. Find the number of degrees in each of the three angles.

26. The cost of repairs in a sawmill was $\frac{3}{16}$ of the total expenses for the month, and the total expenses were $\frac{3}{4}$ of the gross earnings. The net proceeds for the month were \$256. Required the gross earnings and the running expenses.

27. A food ration for a horse contains $\frac{4}{5}$ as much oats as corn and $\frac{1}{3}$ as much mineral-carrying food as other kinds of food. How much of each kind of food is there in 20 lb. of mixed feed?

28. Three postmen deliver mail on routes of different lengths. The first route is 1 mi. shorter than the second and $1\frac{2}{3}$ mi. shorter than the third. The average length of the routes is $4\frac{2}{3}$ mi. Find the length of each route.

29. Three boys went fishing, and on their return they gave this problem: "We caught 19 fish in all, and one of us caught four more than either of the others. How many did each of us catch?"

30. If x, y, and z are three numbers such that $3x = y + 13.6$, $3y = 2z + 6.4$, and twice the sum of the numbers plus the second plus twice the third is 67.8, what are the numbers?

104. Discussion of a Problem. When the result of a problem is expressed in letters as a general formula, the interpretation of the result is called a *discussion of the problem.*

An express train and a freight train are traveling along parallel tracks in the same direction. The express travels e miles an hour and the freight f miles an hour. At noon the freight is d miles in advance of the express. When will the express pass the freight?

If the express passes the freight x hours after noon, it will have traveled ex miles, while the freight will have traveled fx miles. And since the express must gain d miles, we have

$$ex = fx + d;$$

whence

$$x = \frac{d}{e - f}.$$

1. *Suppose e is greater than f* (expressed thus: $e > f$).

Then the denominator is positive and x is positive. This is as it should be, since the express is then traveling faster than the freight and will overtake it *after* noon.

2. *Suppose e is less than f* (expressed thus: $e < f$).

Then x is negative. This is as it should be, since the express is traveling slower than the freight. Therefore if the trains were together at any time, it was *before* noon.

3. *Suppose $e = f$.*

Then $x = \frac{d}{0}$, an expression not yet explained. This fraction may be regarded as a *symbol of infinity* (∞), and written $\frac{d}{0} = \infty$. For if the trains are d miles apart and are traveling at the same rate, they can never be together.

4. *Suppose $e = f$, and $d = 0$.*

Then $x = \frac{0}{0}$. But if the trains are traveling at the same rate and are no distance apart, they are always together. Therefore $\frac{0}{0}$ is to be regarded as a *symbol of indetermination.*

Exercise 59. Discussion of Problems

Examples 1 to 3, oral — Examples 4 to 11, written

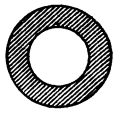
1. If $x = \dfrac{a}{b-c}$, and $b > c$, is x positive or is it negative?

2. In Ex. 1, suppose $b < c$; suppose $b = c$.

3. In Ex. 1, what interpretation is to be given to the result when $a = 0$ and $b = c$?

4. A train traveling r miles an hour is t hours ahead of a second train that travels r' miles an hour. In how many hours will the second train overtake the first? Discuss the problem when $r > r'$; when $r = r'$; when $r < r'$.

5. In the figure here shown, the area of the outer circle is πr_1^2 and that of the inner circle is πr_2^2. Hence the area of the shaded ring is given by the formula $a = \pi(r_1^2 - r_2^2) = \pi(r_1 + r_2)(r_1 - r_2)$. Discuss the problem when $r_1 > r_2$; when $r_1 = r_2$.



6. If a is divided by $b - c$, the result is q. Discuss the nature of q when $b > c$; when $b < c$; when $b = c$; when $a = 0$ and $b = c$; when $a = 0$ and $b > c$ or $b < c$.

7. Two boys live d miles apart. They start from their respective homes at the same time and walk toward each other, one at the rate of a miles an hour, and the other at the rate of b miles an hour. In how many hours after starting will they meet? Discuss the problem when $d = a = b$; when $d = a$ and $b = 0$; when $d = 0$.

8. Discuss the formula $a - b = \dfrac{a^2 - b^2}{a + b}$ when $a > b$; when $a < b$; when $a = b$.

9. Discuss the formula $P = \dfrac{X - Y}{Y - Z}$ when $X > Y > Z$; when $X < Y < Z$; when $X = Y = Z$.

10. Discuss the formula of Ex. 9 when $X = Y$ and $Y > Z$; when $X > Y$ and $Y = Z$.

11. Discuss the formula of Ex. 9 when $X < Y$ and $Y > Z$; when $X = Y$ and $Y < Z$.

CHAPTER VII

POWERS AND ROOTS

105. Power. The product of several equal factors is called a *power* of the factor.

106. Laws of Exponents. The laws of exponents are these:

1. *Products:* $a^m a^n = a^{m+n}$. § 15

2. *Quotients:* $a^m \div a^n = a^{m-n}$. § 19

3. *Powers of products:* $(ab)^m = a^m b^m$.

For $(ab)^m = ab \cdot ab \cdot ab \cdots m$ times
$= (aaa \cdots \text{to } m \text{ factors})(bbb \cdots \text{to } m \text{ factors})$
$= a^m b^m$.

Conversely, $a^m b^m = (ab)^m$.

Similarly, $(abc)^m = a^m b^m c^m$, and so on for any number of factors.

4. *Powers of quotients:* $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

For $\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{to } m \text{ factors}$

$= \frac{aaa \cdots \text{to } m \text{ factors}}{bbb \cdots \text{to } m \text{ factors}}$

$= \frac{a^m}{b^m}$.

Conversely, $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

5. *Powers of powers:* $(a^m)^n = a^{mn}$.

For $(a^m)^n = (aaa \cdots \text{to } m \text{ factors})^n$
$= a^n a^n a^n \cdots \text{to } m \text{ factors, by Law 3}$
$= a^{n+n+n \cdots \text{to } m \text{ terms}}$, by Law 1
$= a^{mn}$.

107. Root. Any one of the equal factors whose product is a given number is called a *root* of the number.

The term is already familiar from arithmetic and has therefore been used whenever necessary in the previous work.

The root is called a square root, cube root, fourth root, or rth root, according as it is one of two, three, four, or r equal factors.

The word *root* is used in two different senses in algebra. The other use is to represent the value of the unknown quantity in an equation. Thus the square root of 25 is 5, and the root of the equation $x - 7 = 23$ is 30.

108. Radical Sign. The common symbol for root $(\sqrt{\ })$ is called the *radical sign.*

Thus, $\sqrt[4]{625} = 5$, because $625 = 5 \cdot 5 \cdot 5 \cdot 5$. In this case 4 is called the *index* of the root. In $\sqrt[n]{a - b}$, n is the index of the root. In the case of a square root, such as $\sqrt{4}$, the index is not written.

109. Imaginary Number. Since any even power of a positive or a negative number is positive, therefore an even root of a negative number cannot be positive and cannot be negative.

We can merely indicate such roots thus: $\sqrt{-1}$, $\sqrt{-3}$, $\sqrt[4]{-2}$. Such numbers are treated later in our work.

An indicated square root of a negative number is called an *imaginary number.*

110. Real Number. A number that does not contain an imaginary number is called a *real number.*

A real number may be an integer (whole number) like 3 or -7, a fraction like $\frac{2}{3}$, $\frac{5}{2}$, $-\frac{5}{8}$, or 0.7, a mixed number like $1\frac{1}{2}$ or -2.75, or such a number as $\sqrt{3}$ or π; but $2 + 3\sqrt{-5}$ is not a real number.

111. Rational Number. An integer or the quotient of two integers is called a *rational number.*

For example, 3, -7, $\frac{2}{3}$, $\frac{5}{2}$, 1.25, and -0.75 are rational numbers.

112. Irrational Number. A real number that is not rational is called an *irrational number.*

For example, $\sqrt{2}$ and $\sqrt{\frac{1}{2}}$ are both irrational. Similarly, numbers like $-2\sqrt{3}$ and $\frac{2}{3}\sqrt[3]{\frac{1}{2}}$ are irrational.

113. Binomial Theorem. By actual multiplication we have

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

In these results it is seen that

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *In the first term the exponent of a is the same as the exponent of the power to which the binomial is raised, and decreases by one in each succeeding term.*

3. *The letter b appears in the second term, with an exponent 1, and the exponent increases by one in each succeeding term.*

4. *The coefficient of the first term is 1, and the coefficient of the second term is the same as the exponent of the binomial.*

5. *The coefficient of each term after the second is found from the preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

The above law is called the *binomial theorem.*

If b is negative, the terms in which the *odd* powers of b occur are *negative.* Thus

$$(a-b)^2 = a^2 - 2ab + b^2,$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$
$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.$$

In these cases it will be noticed that the terms are alternately positive and negative.

The most important part of the Binomial Theorem is 5, and this should be mastered. The theorem is true for any positive integral exponent. Other exponents are considered later in the student's work in mathematics.

A complete proof of the Binomial Theorem is not practicable at this point in our work. Enough has been shown, however, to convince us of the probable truth of the statements above made. A proof of the theorem for positive integral exponents is given on page 234.

1. Raise $2x - 3$ to the second power.

Since $(a - b)^2 = a^2 - 2ab + b^2$,

therefore $(2x - 3)^2 = (2x)^2 - 2 \cdot (2x) \cdot 3 + 3^2$

$= 4x^2 - 12x + 9.$

Check. Let $x = 1$.

Then $(-1)^2 = 1 = 4 - 12 + 9.$

2. Raise $\frac{3}{5}x^2y^3 - \frac{2}{3}$ to the third power.

Since $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,

therefore

$$(\tfrac{3}{5}x^2y^3 - \tfrac{2}{3})^3 = (\tfrac{3}{5}x^2y^3)^3 - 3 \cdot (\tfrac{3}{5}x^2y^3)^2 \cdot \tfrac{2}{3} + 3 \cdot (\tfrac{3}{5}x^2y^3) \cdot (\tfrac{2}{3})^2 - (\tfrac{2}{3})^3$$

$$= \tfrac{27}{125}x^6y^9 - \tfrac{18}{25}x^4y^6 + \tfrac{4}{5}x^2y^3 - \tfrac{8}{27}.$$

Check. Let $x = 1$, $y = 1$.

Then $(\frac{3}{5} - \frac{2}{3})^3 = (-\frac{1}{15})^3 = -\frac{1}{3375}$,

and $\frac{27}{125} - \frac{18}{25} + \frac{4}{5} - \frac{8}{27} = \frac{375}{125} - \frac{8}{27} = -\frac{1}{3375}$.

3. Raise $2x - y^2$ to the fifth power.

Since $(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$,

therefore $(2x - y^2)^5 = (2x)^5 - 5(2x)^4y^2 + 10(2x)^3y^4 - 10(2x)^2y^6$

$+ 5(2x)y^8 - y^{10}$

$= 32x^5 - 80x^4y^2 + 80x^3y^4 - 40x^2y^6 + 10xy^8 - y^{10}.$

Check. Let $x = 1$, $y = 1$.

Then both members of the equation reduce to 1.

Exercise 60. Powers of a Binomial

Examples 1 to 8, oral — Examples 9 to 24, written

1. $(a + 1)^2$.

2. $(a - 2)^2$.

3. $(a + 3)^2$.

4. $(a^2 - b^2)^2$.

5. $(a - 1)^3$.

6. $(a^2 - 1)^3$.

7. $(a^3 + 1)^2$.

8. $(a^3 - 1)^2$.

9. $(4a + 5b)^2$.

10. $(5a^2 - 3b)^2$.

11. $(3a^4 + 2)^2$.

12. $(\frac{2}{3}x - 3y)^3$.

13. $(\frac{3}{4}x^2 + \frac{2}{3}y)^3$.

14. $(\frac{3}{5}x^3 - \frac{1}{2}y^2)^3$.

15. $(0.1x^2 + 0.2y^2)^3$.

16. $(2.1x^2 - 0.3y^2)^3$.

17. $(2a + 1)^4$.

18. $(2m^2 - 3n)^4$.

19. $(\frac{1}{2}p + \frac{2}{3}q)^5$.

20. $(\frac{3}{4}x^2 - \frac{2}{3}y^2)^5$.

21. $(a - 3b)^6$.

22. $(x + 3y^2)^6$.

23. $(2x + y^3)^7$.

24. $(1.1x + 2.1y)^4$.

114. Signs of Roots. Since $(+2)(+2)=4$, and $(-2)(-2)=4$, we see that 4 has two square roots, $+2$ and -2. For the same reason it is evident that

Every positive number has two real square roots which have the same absolute values but have opposite signs.

For a similar reason every number has at least two real fourth roots, two real sixth roots, and so on for any roots indicated by an even number.

For example, 9 has two real square roots, $+3$ and -3; and 16 has two fourth roots, $+2$ and -2, since $(+2)^4=16$, and $(-2)^4=16$. Imaginary roots are discussed later.

As a matter of fact, every number has three cube roots, four fourth roots, five fifth roots, and so on. Some of these roots are imaginary, but we do not need to consider such roots at the present time.

Furthermore, since $(+2)^3=+8$, and $(-2)^3=-8$, we see that $\sqrt[3]{+8}=+2$, and $\sqrt[3]{-8}=-2$. For the same reason it is evident that

An odd root of a number has the same sign as the number itself.

For example, because $(-2)^5=-32$, therefore $\sqrt[5]{-32}=-2$, while $\sqrt[5]{+32}=+2$.

115. Principal Root. The one real root of a number, if it has but one, or its real positive root, if it has two real roots, is called the *principal root* of the number.

Thus the principal square root of 4 is $+2$, although there is another root, -2; and the principal cube root of -27 is -3, although it will later be shown that there are two imaginary roots.

116. Radical Sign and Principal Root. It is to be understood in algebra that *the radical sign indicates the principal root.*

Thus $\sqrt{4}=2$, and it is incorrect to write $\sqrt{4}=-2$, although 4 has two square roots. If we wish to indicate the negative root, we write $-\sqrt{4}=-2$, and if we wish to indicate both roots, we write $\pm\sqrt{4}=\pm 2$.

Therefore if we have to simplify $\sqrt{4}+\sqrt{9}+\sqrt{16}$, we have $2+3+4=9$, and not $\pm 2\pm 3\pm 4$, which has several possible values.

Similarly, if we have the equation $x^2=5$, we should write $x=\pm\sqrt{5}$ and not merely $x=\sqrt{5}$.

117. Square Root of a Polynomial. If we square $a + b$ the result is, as shown in § 24, $a^2 + 2ab + b^2$. In extracting the square root of $a^2 + 2ab + b^2$ we reverse the process of squaring, thus:

The given square	$= a^2 + 2ab + b^2$	$\vert a + b$	$=$ Root
The square of a	$= a^2$		
Trial divisor	$= 2a$	$\vert 2ab + b^2$	$=$ First remainder
Complete divisor	$= 2a + b$	$\vert 2ab + b^2$	$= b(2a + b)$

The first term of the root is evidently a, because the square of a is a^2.

Since in squaring a binomial we have the square of the first term plus twice the product of the first and second terms, etc., we have in $2ab$ twice the product of a and the second term. We therefore divide by $2a$ to find the second term. Since $2ab \div 2a$ equals b, b is the second term.

In squaring $a + b$ we have $a^2 + b(2a + b)$. We therefore add b to $2a$, and multiply $2a + b$ by b, thus completing the square of $a + b$.

The names *trial divisor* and *complete divisor* are used as above shown, being convenient names brought into algebra from arithmetic.

Find the square root of $4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6$.

$4x^4 - 12x^2y + 9y^2 + 4x^2z^3 - 6yz^3 + z^6$	$\vert 2x^2 - 3y + z^3$	
$4x^4$		
$4x^2$	$-12x^2y + 9y^2$	
$4x^2 - 3y$	$-12x^2y + 9y^2$	
$4x^2 - 6y$		$4x^2z^3 - 6yz^3 + z^6$
$4x^2 - 6y + z^3$		$4x^2z^3 - 6yz^3 + z^6$

The first term of the root is $2x^2$.

Subtracting the square of $2x^2$, which is $4x^4$, the remainder contains twice the first term times the second, plus the square of the second. Dividing by $4x^2$, which is twice $2x^2$, the second term is $-3y$.

Proceeding as before and subtracting, we may consider the second remainder as twice the product of $(2x^2 - 3y)$ times the next term, considering the binomial $2x^2 - 3y$ as the first term.

Dividing by twice $2x^2 - 3y$, or by $4x^2 - 6y$, we find the next term to be z^3.

Proceeding as before and subtracting, there is no remainder. Hence the square root is $2x^2 - 3y + z^3$.

We may check this result by squaring it, or we may let $x = 2$, $y = 1$, $z = 1$ (or any other convenient values).

118. Square Root of a Polynomial containing Fractions. If a polynomial contains powers and reciprocals of the same letter, the order of arrangement in descending powers of the letter is as follows:

$$\cdots x^4, x^3, x^2, x, 1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \cdots$$

Find the square root of $x^2 + 13 + 6x + \frac{4}{x^2} + \frac{12}{x}$.

Arrange according to the descending powers of x, and proceed as before.

$$\begin{array}{r|l} x^2 + 6x + 13 + \frac{12}{x} + \frac{4}{x^2} & x + 3 + \frac{2}{x} \\ x^2 & \\ \hline 2x \quad | \; 6x + 13 & \\ 2x + 3 \; | \; 6x + 9 & \\ \hline 2x + 6 \quad | \; 4 + \frac{12}{x} + \frac{4}{x^2} & \\ 2x + 6 + \frac{2}{x} \; | \; 4 + \frac{12}{x} + \frac{4}{x^2} & \end{array}$$

Check. Let $x = 2$, so as to avoid fractions. Then

$$4 + 12 + 13 + 6 + 1 = 36,$$

and

$$2 + 3 + 1 = 6 = \sqrt{36}.$$

If the terms of the given polynomial are not arranged according to the ascending or descending powers of some letter, make this arrangement before beginning the work.

119. Extracting the Square Root. Therefore, in extracting the square root of a polynomial,

Arrange the terms according to the powers of some letter.

Find the principal square root of the first term, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the first term of the root, and write the quotient as the second term of the root.

Multiply the sum of twice the first term and the second term of the root by the second term, thus completing the square of the sum of the first two terms, and subtract this product.

Consider what has now been found as the first part of the root, and proceed as before.

Exercise 61. Square Root of a Polynomial

Examples 1 to 4, oral — Examples 5 to 17, written

1. Find the square root of $x^2 + 2 + \frac{1}{x^2}$; of $4x^2 + 4 + \frac{1}{x^2}$.

2. What are the two square roots of $x^2 - 6 + \frac{9}{x^2}$?

Find the square roots of the following:

3. $\frac{x^2}{9} + \frac{2x}{3} + 1$.

4. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2}$.

5. $25a^4 + \frac{9}{b^2} + \frac{4}{c^4} + \frac{30a^2}{b} - \frac{20a^2}{c^2} - \frac{12}{bc^2}$.

6. $\frac{x^2}{4z^4} + \frac{4z^2}{25x^4} + \frac{2}{5xz} - \frac{12z}{5x^2} - \frac{3x}{z^2} + 9$.

7. $\frac{x^2}{y^2} - \frac{4xz}{ay} + \frac{4z^2}{a^2} + \frac{6qx}{by} + \frac{9q^2}{b^2} - \frac{12qz}{ab}$.

8. $x^6y^6 - \frac{2x^3y^{12}}{3a^3} + \frac{y^{18}}{9a^6} + \frac{4x^3y^{18}}{a^9} - \frac{4y^{24}}{3a^{12}} + \frac{4y^{30}}{a^{18}}$.

9. $\frac{9x^2y^2}{49a^2b^2} - \frac{15xy}{28ab} + \frac{203}{192} - \frac{35ab}{36xy} + \frac{49a^2b^2}{81x^2y^2}$.

10. $\frac{1}{4}p^4 - \frac{1}{3}p^2q^3 + \frac{1}{9}q^6 + \frac{1}{4}p^2r^4 + \frac{1}{18}r^8 - \frac{1}{6}q^3r^4$.

11. $\frac{1}{4}m^4 + \frac{1}{3}m^3n - \frac{5}{36}m^2n^2 - \frac{1}{6}mn^3 + \frac{1}{16}n^4$.

12. $4a^6 - 14a^2b^4 + 25b^5 - 7a^4b^2 + 12a^5b + 40ab^5 - 44a^3b^3$.

13. $4p^6 - 20p^5q - 3p^4q^2 + 9q^6 - 42pq^5 + 19p^2q^4 + 82p^3q^3$.

14. $a^2 + 2ab + b^2 + 2ac + 2bc + c^2 + 2ad + 2bd + 2cd + d^2$.

15. If the square root of $a^4 + 8a^3 + 10a^2 - 24a + 9$ is $a^2 + xa - y$, what are the values of x and y?

16. If the square root of $a^2 + xa + 9$ is $a + y$, what are the values of x and y?

17. It is known that $x^4 + 6x^3 + 19x^2 + 30x + 25$ is an exact square of a trinomial. Without going through the complete process of extracting the square root, state the first term of the root; the last term of the root; the second term of the root; the entire root. Check the result.

120. Arithmetical Square Root. The first step in extracting the square root of a number is to separate the figures of the number into groups of two figures each, called *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any integral number expressed by *one* or *two* figures is a number of *one* figure; expressed by *three* or *four* figures is a number of *two* figures; and so on.

Therefore if an integral number is separated into periods of two figures each, from the right to the left, the number of figures in the square root will be equal to the number of the periods of figures. The last period at the left may have two figures or one figure; for example, 22 09, and 7 89 04 81.

1. Find the square root of 2209.

We first recall that if $t =$ tens and $u =$ units, we have $(t + u)^2 = t^2 + 2tu + u^2$.

Separating into periods, we see that there will be two integral places in the root.

The first period, 22, contains t^2. And since the greatest square in 22 is 16, therefore $\sqrt{16}$, or 4, is the tens' figure of the root, or t.

Subtracting t^2, the remainder contains $2tu + u^2$. Therefore if we divide by $2t$ (that is, by 80, which is 2×4 tens), we shall find approximately u, the units. Dividing 609 by 80 (or 60 by 8), we have 7 as the units' figure.

```
 22 09 (47
 16
---
80  6 09
87  6 09
---
```

Since $2tu + u^2 = (2t + u)u$, that is, $2 \times 40 \times 7 + 7^2 = (2 \times 40 + 7) \times 7$, we add 7 to 80 and multiply the sum by 7. The product is 609, thus completing the square of 47.

2. Find the square root of 7,890,481.

When the third period, 04, is brought down, and the divisor, 560, formed, the next figure of the root is 0, because 560 is not contained in 504. Therefore 0 is placed in the root, another 0 is annexed to the divisor, and the next two figures, 81, are brought down.

```
  7 89 04 81 (2809
  4
---
40  3 89
48  3 84
---
5600    5 04 81
5609    5 04 81
---
```

The work then proceeds as before, the trial divisor now being 5600, which is twice 280 tens. The entire root is thus found to be 2809.

121. Approximate Square Roots. If a number is not a perfect square, zeros may be annexed and an approximate value of the root found.

For example, extract to three places of decimals the square root of 19.

	19.00 00 00 (4.3588 +
	16
80	3 00
83	2 49
860	51 00
865	43 25
8700	7 75 00
8708	6 96 64
87160	78 3600

In this example we proceed in the usual way, annexing pairs of zeros for each decimal place in the root, and separating into periods of two figures each to the right and left of the decimal point. We carry the work to four decimal places, so as to find the nearest approximation for three places. The result is found to be 4.359 — ; that is, it is nearer 4.359 than 4.358.

122. Summary of Square Root. We may summarize the process of square root as follows:

Separate the number into periods of two figures each, beginning at the decimal point.

Find the greatest square in the period at the left and write its root for the first figure of the required root.

Square this root, subtract the result from the period at the left, and to the remainder annex the next period for a dividend.

For a partial divisor double the root already found, considered as tens, and divide the dividend by it. The quotient (or the quotient diminished slightly if necessary) will be the next figure of the root.

To the partial divisor add this next figure of the root for a complete divisor. Multiply the complete divisor by this next figure of the root, subtract the product from the dividend, and to the remainder annex the next period for a new dividend.

Proceed in this manner until all the periods have been thus annexed. The result will be the square root required.

123. Square Roots of Common Fractions. To find the square root of a common fraction we may extract the square roots of the numerator and denominator.

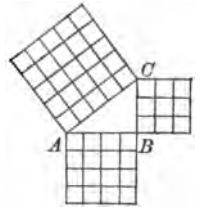
If the denominator is not a perfect square, it is better to multiply both terms of the fraction by a number that will make it such, or else to reduce to a decimal and then extract the square root. Thus, instead of saying $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{1.414+}$, which requires us to divide the numerator by a long divisor, it is better to say $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2} = \frac{1}{2}$ of $1.414+ = 0.707+$, or else to extract the square root of 0.5.

124. Right Triangle. A triangle that contains a right angle is called a *right triangle.* It is sometimes called a *right-angled triangle.*

Certain definitions of this kind, familiar from the work in arithmetic and in Book I, are reviewed from time to time because of their relation to the particular topic under discussion.

125. Hypotenuse. In a right triangle the side opposite the right angle is called the *hypotenuse.*

In the triangle ABC the hypotenuse is AC, the base is AB, and the perpendicular (or altitude) is BC.



126. Square on the Hypotenuse. As already stated on page 8, by examining this figure we see that the square on AC equals the sum of the squares on AB and BC. It is proved in geometry that this is true for all right triangles.

Therefore, *in a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides.*

The hypotenuse equals the square root of the sum of the squares of the other two sides.

The base or the perpendicular equals the square root of the difference of the squares of the hypotenuse and the other side.

This is one of the most important propositions of geometry. It is said to have been first proved in the sixth century B.C. by the Greek mathematician Pythagoras, a native of Samos, in the Ægean Sea.

127. Applications of Square Root. It is proved in geometry that

The corresponding lines of similar figures are to each other as the square roots of the areas of the figures.

That is, if one circle has 16 times the area of another, the radius is 4 times as long as that of the other.

The radius of a circle equals approximately the square root of the quotient of the area of the circle divided by 3.1416.

For example, if the area of a circle is 78.54 sq. in., the number of inches in the radius is $\sqrt{\frac{78.54}{3.1416}} = \sqrt{25} = 5$. Therefore the radius is 5 in.

In Exercise 62 carry the square roots to two decimal places.

Exercise 62. Square Root

Find the square root of:

1. 3249.	7. 592,900.	13. 7.	19. $\frac{1}{5}$.
2. 41.2164.	8. 156,816.	14. 29.	20. $\frac{1}{3}$.
3. 48.1636.	9. 426,888.	15. 415.	21. $\frac{1}{8}$.
4. 84,681.	10. 3,415,104.	16. 0.24373969.	22. $\frac{1}{2}$.
5. 965.9664.	11. 3,111,696.	17. 4.54499761.	23. $\frac{2}{3}$.
6. $\frac{100}{144}$.	12. $\frac{169}{225}$.	18. $\frac{289}{324}$.	24. $\frac{3}{4}$.

Find the radius of the circle whose area is:

25. 628.32 sq. in. 26. 113.0976 sq. in. 27. 500 sq. in.

28. What must be the diameter of a water main to have the area of a cross section $3\frac{1}{2}$ sq. ft.?

29. A horse tethered by a rope can graze over 1570.8 sq. ft. of ground. How long is the rope? If the rope were twice as long, over how much ground could he graze?

30. A tinsmith wishes to make some cylindrical gallon cans. They are to be 10 in. high. What must be the area of the base? What radius must he use to draw it? (1 gal. = 231 cu. in.)

128. Cube Root of a Polynomial. If we cube $a + b$, the result is, as shown in § 24; $a^3 + 3a^2b + 3ab^2 + b^3$. In extracting the cube root we reverse this process of cubing, thus:

$$a^3 + 3a^2b + 3ab^2 + b^3 \mid a + b = \text{Root}$$

$$a^3$$

$3a^2$	$3a^2b + 3ab^2 + b^3 =$ First remainder
$3a^2 + 3ab + b^2$	$3a^2b + 3ab^2 + b^3 = b(3a^2 + 3ab + b^2)$

The first term of the root is evidently a, because the cube of a is a^3.

Since in cubing a binomial we have the cube of the first term plus three times the product of the square of the first by the second, etc., we have in $3a^2b$ three times the product of the square of a by the second term. We therefore divide by $3a^2$ to find the second term, b.

In cubing $a + b$ we have $a^3 + b(3a^2 + 3ab + b^2)$. We therefore add $3ab + b^2$ to $3a^2$ and multiply $3a^2 + 3ab + b^2$ by b, thus completing the cube of $a + b$.

1. Find the cube root of $27x^3 - 189x^2y + 441xy^2 - 343y^3$.

$$27x^3 - 189x^2y + 441xy^2 - 343y^3 \mid 3x - 7y$$

$$27x^3$$

$27x^2$	$-189x^2y + 441xy^2 - 343y^3$
$27x^2 - 63xy + 49y^2$	$-189x^2y + 441xy^2 - 343y^3$

Here $3(3x)^2 = 27x^2$, corresponding to $3a^2$ in $(a + b)^3$.

To $27x^2$ is added $3(3x)(-7y) + (-7y)^2$, or $-63xy + 49y^2$.

This work should be compared, step by step, with the example above explained, the following questions being answered:

How is the first term of the root found?

Why is $27x^2$ taken as the divisor?

How is the second term found?

Why is $-63xy + 49y^2$ added to $27x^2$?

Why is $27x^2 - 63xy + 49y^2$ multiplied by $-7y$?

2. Find the cube root of $300x^2y - 5x(25x^2 + 48y^2) + 64y^3$. Rearranging, the work appears as follows:

$$-125x^3 + 300x^2y - 240xy^2 + 64y^3 \mid -5x + 4y$$

$$-125x^3$$

$75x^2$	$300x^2y - 240xy^2 + 64y^3$
$75x^2 - 60xy + 16y^2$	$300x^2y - 240xy^2 + 64y^3$

In case there are more than two terms in the root the method is essentially the same. After two terms have been found, their sum is considered to be a in the typical form $a^3 + 3a^2b + 3ab^2 + b^3$.

Thus if the part of the root already found is $x^2 - x$, we subtract $(x^2 - x)^3$, and the remainder is in the form $3a^2b + 3ab^2 + b^3$, in which $x^2 - x$ is a. We therefore divide by $3(x^2 - x)^2$ to find b, the next term. We proceed in a similar manner if there are more than three terms in the root, always letting a represent the part of the root already found.

3. Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$
\begin{array}{l|l}
x^6 - 3x^5 + 5x^3 - 3x - 1 & x^2 - x - 1 \\
x^6 & \\
\hline
3x^4 & -3x^5 \quad +5x^3 \\
3x^4 - 3x^3 + x^2 & -3x^5 + 3x^4 - x^3 \\
\hline
3x^4 - 6x^3 + 3x^2 & -3x^4 + 6x^3 - 3x - 1 \\
3x^4 - 6x^3 + 3x + 1 & -3x^4 + 6x^3 - 3x - 1 \\
\hline
\end{array}
$$

The first term of the root is x^2.

The first divisor is $3(x^2)^2$, or $3x^4$; and $-3x^5 \div 3x^4 = -x$, the second term.

Then $3(x^2)^2 + 3(x^2)(-x) + (-x)^2 = 3x^4 - 3x^3 + x^2$. This, multiplied by $-x$, equals $-3x^5 + 3x^4 - x^3$, thus completing the cube of $x^2 - x$.

Subtracting, and considering $x^2 - x$ as the first part of the root, a, we have subtracted the cube of the first part of the root, a^3. The remainder therefore contains $3a^2b + 3ab^2 + b^3$, where b is the next term. We therefore divide this remainder by $3a^2$, or $3(x^2 - x)^2$. Practically we need divide only $-3x^4$ by $3x^4$. We therefore find that the next term is -1.

As before, we multiply $3a^2 + 3ab + b^2$ by b, or $3x^4 - 6x^3 + 3x + 1$ by -1, thus completing the cube of $x^2 - x - 1$.

4. Find the cube root of $x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$.

Leaving the divisors to be found by the student, we have

$$
\begin{array}{l|l}
x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 & x^2 + 2x + 1 \\
x^6 & \\
\hline
6x^5 + 15x^4 + 20x^3 & \\
6x^5 + 12x^4 + 8x^3 & \\
\hline
3x^4 + 12x^3 + 15x^2 + 6x + 1 & \\
3x^4 + 12x^3 + 15x^2 + 6x + 1 & \\
\hline
\end{array}
$$

Exercise 63. Cube Root of Polynomials

Examples 1 to 4, oral — Examples 5 to 21, written

1. What is the cube root of $x^3 + 3x^2 + 3x + 1$?

2. What is the cube root of $x^3 + 6x^2 + 12x + 8$?

3. What is the cube root of $27 - 27x + 9x^2 - x^3$?

4. What are the first and the last terms of the cube root of the perfect cube $125x^9 + 300x^8 + 465x^7 + 424x^6 + 279x^5 + 108x^4 + 27x^3$?

Find the cube roots of the following polynomials:

5. $27x^3 - 54x^2y + 36xy^2 - 8y^3$.

6. $m^6 - 3m^4n + 3m^2n^2 - n^3$.

7. $125a^3 + 225a^2b + 135ab^2 + 27b^3$.

8. $343n^3 + 735n^2m + 125m^3 + 525m^2n$.

9. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$.

10. $8a^6 - 60a^5 + 114a^4 + 55a^3 - 171a^2 - 135a - 27$.

11. $64x^6 - 144x^5y + 60x^4y^2 - y^6 - 9xy^5 - 15x^2y^4 + 45x^3y^3$.

12. $27n^6 + 189n^5 + 198n^4 - 791n^3 - 594n^2 + 1701n - 729$.

13. $x^3 + 9x^2y + 27xy^2 + 27y^3$.

14. $64m^3 - 144m^2n + 108mn^2 - 27n^3$.

15. $27x^9 - 135x^8 + 198x^7 - 35x^6 - 66x^5 - 15x^4 - x^3$.

16. $27x^6 + 108x^5 + 198x^4 + 208x^3 + 132x^2 + 48x + 8$.

17. $8x^6 - 12x^5 + 30x^4 - 25x^3 + 30x^2 - 12x + 8$.

18. $\frac{8a^3}{27x^3} + \frac{4a^2}{3x^2} + \frac{3a}{x} + 4 + \frac{27x}{8a} + \frac{27x^2}{16a^2} + \frac{27x^3}{64a^3}$.

19. $8a^3 + 36a^2b - 12a^2c + 27b^3 + 54ab^2 + 6ac^2 - 27b^2c + 9bc^2 - c^3 - 36abc$.

20. $64x^{12} - 144x^{10} + 96x^9 + 108x^8 - 144x^7 + 21x^6 + 54x^5 - 36x^4 + 8x^3$.

21. Write out a rule for cube root similar to that for square root on page 132.

129. Cube Root of Numbers. The first step in extracting the cube root of a number is to separate the figures of the number into groups of three figures each, called *periods*.

Since $1 = 1^3$, $1000 = 10^3$, $1{,}000{,}000 = 100^3$, and so on, the cube root of any integral number that has *one*, *two*, or *three* figures is a number of *one* figure; the cube root of any integral number that has *four*, *five*, or *six* figures is a number of *two* figures; and so on.

If, therefore, an integral number is separated into periods of three figures each, from right to left, the number of figures in the root will be equal to the number of periods. The last period to the left may consist of one figure, two figures, or three figures.

For example, the cube root of 34,645,976 will have three integral places.

Cube roots are required in practical work in engineering, but such roots are usually found by tables, logarithms, or some such mechanical device as the slide rule or the computing machine. In order to use the computing machine, as much knowledge of the theory is needed as is here set forth.

Find the cube root of 42,875.

We first recall that if $t =$ tens and $u =$ units, we have $(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3$.

We see that there will be two integral places in the root. The first period, 42, contains t^3. The greatest cube in 42 is 27, and the cube root of 27 is 3. Hence $t = 3$.

$$\begin{array}{rrl} & 42\ 875 & (35 \\ & 27 & \\ \hline 3 \times 30^2 = & 2700 & 15\ 875 \\ 3 \times (30 \times 5) = & 450 & \\ 5^2 = & 25 & \\ \hline & 3175 & 15\ 875 \end{array}$$

The remainder, 15,875, resulting from subtracting the cube of the tens, will contain $3t^2u + 3tu^2 + u^3$.

Each of these three parts contains u as a factor.

Hence the 15,875 consists of two factors, one of which is u, and the other is $3t^2 + 3tu + u^2$. The largest part of this second factor is $3t^2$.

If the 158 hundreds of the remainder is divided by $3t^2 = 3 \times 30^2$, or 27 hundreds, the quotient will be approximately u. The second factor can now be completed by adding to the 2700 the sum of $3 \times (30 \times 5)$, or 450, and 5^2, or 25.

If this factor, 3175, is now multiplied by 5, the result is 15,875, which completes the cube of 35. There being no remainder, $\sqrt[3]{42875} = 35$.

To check the work, $35^3 = 42{,}875$.

130. Cube Root of Larger Numbers. The method of § 129 can be applied to numbers of more than two periods, by considering *the part of the root already found as so many tens* with respect to the next figure of the root.

For example, find the cube root of 57,512,456.

		57 512 456(386
		27
$3 \times 30^2 =$	2700	30 512
$3 \times (30 \times 8) =$	720	
$8^2 =$	64	
	3484	27 872
$3 \times 380^2 =$	433200	2 640 456
$3 \times (380 \times 6) =$	6840	
$6^2 =$	36	
	440076	2 640 456

Therefore the cube root is 386.

131. Cube Root of Decimals. If a cube root has decimal places, the cube will have *three times* as many.

Thus, if 0.11 is the cube root of a number, the number is $0.11 \times 0.11 \times 0.11 = 0.001331$. Hence, if a given number contains a decimal, we separate it into periods of three figures each, beginning at the decimal point and proceeding toward the left for the integral part, and toward the right for the decimal. The last period of the decimal must contain *three* figures, zeros being annexed when necessary.

Find the cube root of 187.149248.

		187.149 248(5.72
		125
$3 \times 50^2 =$	7500	62 149
$3 \times (50 \times 7) =$	1050	
$7^2 =$	49	
	8599	60 193
$3 \times 570^2 =$	974700	1 956 248
$3 \times (570 \times 2) =$	3420	
$2^2 =$	4	
	978124	1 956 248

Since there can be only one integral place, the decimal point is placed after the 5, the cube root being 5.72.

132. Approximate Cube Root. If the given number is not a perfect cube, zeros may be annexed and a value of the root may be found as near to the true value as we please.

Extract the cube root of 1250.6894.

$$\begin{array}{rrr|l} & & & 1\ 250.689\ 400\,(10.77 \\ & & & 1 \\ \hline 3\times 10^2 = & 300 & 250 & \end{array}$$

Since 300 is not contained in 250, the next figure of the root is 0.

$$\begin{array}{rrr|l} 3\times 100^2 = & 30000 & 250\ 689 & \\ 3\times(100\times 7) = & 2100 & & \\ 7^2 = & 49 & & \\ \hline & 32149 & 225\ 043 & \\ \hline 3\times 1070^2 = & 3434700 & 25\ 646\ 400 & \\ 3\times(1070\times 7) = & 22470 & & \\ 7^2 = & 49 & & \\ \hline & 3457219 & 24\ 200\ 533 & \\ \hline & & 1\ 445\ 867 & \end{array}$$

Exercise 64. Cube Root of Numbers

Extract the cube roots of the following numbers:

1. 1,771,561.	**4.** 47,832,147.	**7.** 4826.809.
2. 1,295,029.	**5.** 11,390,625.	**8.** 0.000912673.
3. 2,048,383.	**6.** 87,528,384.	**9.** 0.114791256.

Find the edges of the cubes having the following volumes:

10. 75 cu. in. **11.** 830 cu. ft. **12.** 92.5 cu. in. **13.** $7\frac{1}{2}$ cu. in.

14. Find the diameter of an iron ball that weighs 27 times as much as an iron ball 2 in. in diameter.

15. The weights of two iron cylinders of the same shape are as 2197 to 4913. Find the ratio of their heights.

16. Find the edge of a cube whose volume is equal to the volume of a rectangular solid $81'' \times 3'' \times 3''$.

17. Find the edge of a cubical cistern that holds as many gallons as a rectangular cistern $12' \times 8' \times 5\frac{1}{3}'$.

133. The *n*th Power of the *n*th Root. Because $\sqrt{3}$ means one of the two equal factors of 3, $\sqrt{3} \cdot \sqrt{3}$ must equal 3. That is, $(\sqrt{3})^2 = 3$. By the same reasoning

The nth power of the nth root of a number equals the number itself.

That is, $$(\sqrt[n]{a})^n = a.$$

For a similar reason, *the nth root of the nth power of a number equals the number itself.*

That is, $$\sqrt[n]{a^n} = a.$$

134. The *n*th Root of a Product. Because it has already been proved that

$$ab = (\sqrt{a})^2(\sqrt{b})^2, \qquad \S\ 133$$

$$= (\sqrt{a} \cdot \sqrt{b})^2, \qquad \S\ 106$$

therefore $$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

By the same reasoning

The nth root of the product of two or more numbers equals the product of their nth roots.

That is, $$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Reversing this statement, $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$. That is, $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, $\sqrt[3]{4} \cdot \sqrt[3]{3} = \sqrt[3]{12}$, etc.

135. The *n*th Root of a Power. Since we know that

$$\sqrt[3]{a^6} = \sqrt[3]{a^3 a^3} = aa = a^2, \qquad \S\ 134$$

and that $$\sqrt[n]{a^{2n}} = \sqrt[n]{a^n a^n} = aa = a^2,$$

Therefore, *the exponent of the nth root of any power of a number is found by dividing the exponent of the power by the index of the root.*

That is, $$\sqrt[3]{x^{12}} = x^4, \quad \sqrt[5]{n^{10}} = n^2.$$

Similarly, $$\sqrt[n]{a^{mn}} = a^m.$$

If the exponent is not exactly divisible by the index of the root, we proceed as follows:

$$\sqrt[5]{n^{11}m^{20}} = \sqrt[5]{n^{10}m^{20}n} = n^2m^4\sqrt[5]{n}.$$

136. Surd. If an indicated root of a positive rational number cannot be obtained exactly, the indicated root is called a *surd.*

For example, $\sqrt{2}$ and $\sqrt[3]{\frac{2}{3}}$ are surds, but any expression in which the radical sign enters is called a *radical*, such as $a + \sqrt{b}$.

137. Surds classified. Surds are classified according to the index of the root, being called *quadratic, cubic,* and *biquadratic,* according as this index is 2, 3, or 4.

We might have other names, as *quintic* (fifth root), *sextic* (sixth root), and so on, but it is not necessary to use these terms.

Surds are also classified as *mixed surds,* when there is a rational coefficient; and as *entire surds,* when there is no rational coefficient other than 1.

Thus $2\sqrt{3}$ is a mixed surd, and $\sqrt[3]{7}$ is an entire surd.

138. Reduction of Surds. The changing of the form of a surd without changing its value is called a *reduction* of the surd.

Thus it will be seen that $\sqrt{4a} = 2\sqrt{a}$, and in this case we have reduced $\sqrt{4a}$ to another form.

139. Surds in Simplest Form. When the quantity under the radical sign is integral and as small as possible, a surd is said to be in its *simplest form.*

That is, an integer left under the radical sign must be as small as possible, a letter must have as small an exponent as possible, and there must be no fraction left under the radical sign.

For example, $\sqrt{4a}$ is not in its simplest form, for we can extract the square root of 4, nor is $\sqrt[3]{2a^3}$ in its simplest form.

140. Illustrative Problems. Consider the following problems:

1. Simplify $\sqrt[3]{a^3}$, $(\sqrt[3]{a})^3$, and $\sqrt[n]{a^n}$.

$\sqrt[3]{a^3} = a$, $(\sqrt[3]{a})^3 = a$, and $\sqrt[n]{a^n} = a$, by § 133.

2. Simplify $\sqrt{25\,a^2b^8}$, $\sqrt[3]{27\,a^6b^{12}}$, and $\sqrt[5]{32\,x^{10}}$.

$\sqrt{25\,a^2b^8} = 5\,ab^4$, $\sqrt[3]{27\,a^6b^{12}} = 3\,a^2b^4$, and $\sqrt[5]{32\,x^{10}} = 2\,x^2$, by §§ 134 and 135.

3. Simplify $\sqrt[5]{243\,a^6x^{10}}$.

$$\sqrt[5]{243\,a^6x^{10}} = \sqrt[5]{3^5a^5x^{10}a} = 3\,ax^2\sqrt[5]{a}, \text{ by §§ 134 and 135.}$$

Exercise 65. Simplifying Surds

Simplify the following:

1. $(\sqrt{7})^2$.
2. $(\sqrt[4]{7})^4$.
3. $(\sqrt[5]{5})^5$.
4. $(\sqrt[3]{5})^3$.
5. $\sqrt{4a^2}$.
6. $\sqrt{9a^2b^4}$.
7. $\sqrt{25x^4y^6}$.
8. $\sqrt{36x^6y^8}$.
9. $\sqrt{49x^8y^{10}}$.
10. $\sqrt{81m^2n^{12}}$.
11. $\sqrt{625p^4q^{10}}$.
12. $\sqrt{169a^6b^6c^2}$.
13. $\sqrt[3]{27a^6c^9}$.
14. $\sqrt[3]{64x^{12}y^{15}}$.
15. $\sqrt[3]{125m^{15}n^{18}}$.
16. $\sqrt[4]{16a^4b^8c^{12}}$.
17. $\sqrt[4]{81m^8n^{16}}$.
18. $\sqrt[5]{32a^{10}b^{15}}$.
19. $\sqrt[5]{243x^{15}y^{20}}$.
20. $\sqrt[6]{64x^{12}y^{18}}$.
21. $\sqrt[7]{2187a^7b^{14}}$.
22. $\sqrt[8]{256a^{16}b^{24}}$.
23. $\sqrt[9]{512a^9b^{18}}$.
24. $\sqrt[10]{1024a^{10}b^{40}}$.
25. $\sqrt[n]{a^nb^{2n}c^{3n}}$.
26. $\sqrt{2a^2}$.
27. $\sqrt{8a^4b^2}$.
28. $\sqrt{9ab^2}$.
29. $\sqrt{16a^4b}$.
30. $\sqrt{50a^4b}$.
31. $\sqrt{144abc}$.
32. $\sqrt{49a^4b^2c}$.
33. $\sqrt{a^{10}b^8c^6}$.
34. $\sqrt{a^{11}b^9c^7}$.
35. $\sqrt{a^{11}b^{10}c^8}$.
36. $\sqrt{(a+b)^2c}$.
37. $\sqrt{(a+b)c^2}$.
38. $\sqrt[3]{-27a^4b^4}$.
39. $\sqrt[3]{-125x^5y^8}$.
40. $\sqrt[3]{-16x^7y^{10}}$.
41. $\sqrt[4]{32a^4}$.
42. $\sqrt[4]{32a^5b}$.
43. $\sqrt[4]{64a^{10}b^{10}}$.
44. $\sqrt[5]{-32a^5b^5}$.
45. $\sqrt[5]{-64a^5b^5}$.
46. $\sqrt[6]{2a^{12}b^6}$.
47. $\sqrt[7]{-128x^7y^{14}}$.
48. $\sqrt[8]{2m^8n^9}$.
49. $\sqrt[9]{-512x^9y^{10}}$.
50. $\sqrt[10]{a^{11}b^{11}c^{11}d}$.
62. $\sqrt[5]{-128a^{25m}}$.
63. $\sqrt[6]{128x^{6m}}$.
64. $\sqrt[7]{a^7b^{7m}c^8}$.
65. $\sqrt[7]{a^{7m+1}b^{7m+2}}$.
66. $\sqrt[8]{x^8y^{8a}z^{8b}}$.
67. $\sqrt[9]{-x^9y^{9m}}$.
68. $\sqrt[9]{-x^{10}y^{9m+2}}$.
69. $\sqrt[10]{a^{20}b^{21}c^{28}}$.
70. $\sqrt[11]{a^{22}b^{28}c^{24}}$.
71. $\sqrt[n]{ab^2c^3d^{4n}}$.
72. $\sqrt[n]{a^nb^{n+2}c^{n+3}}$.
73. $\sqrt[n]{a^na^2b^nb^2}$.
74. $\sqrt[n]{(a+b)^nc^{2n}}$.
75. $\sqrt[n]{(a+b+c)^{4n}}$.

141. Reduction of a Mixed Surd to an Entire Surd. Since we have found that $a = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6}$,

we see that $a^2\sqrt[3]{b} = \sqrt[3]{a^6} \cdot \sqrt[3]{b} = \sqrt[3]{a^6 b}$.

Therefore, *to reduce a mixed surd to an entire surd, raise the coefficient of the surd to the power indicated by the index of the radical, and introduce it as a factor under the radical sign.*

That is, $a\sqrt[n]{b} = \sqrt[n]{a^n b}$.

142. The *n*th Root of a Quotient. If we square $\frac{\sqrt{2}}{\sqrt{3}}$ and $\sqrt{\frac{2}{3}}$ we obtain the same result, thus:

$$\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2 = \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{3}\cdot\sqrt{3}} = \frac{2}{3}, \quad \text{and} \quad \left(\sqrt{\frac{2}{3}}\right)^2 = \frac{2}{3}. \qquad \S\ 133$$

The principal nth root of the quotient of two numbers equals the quotient of their principal nth roots.

Stated in symbols, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Exercise 66. Reduction of Surds

Reduce to entire surds:

1. $ab\sqrt{ab}$.
2. $a^2b^2\sqrt{ab}$.
3. $4a\sqrt{a+b}$.
4. $-3a\sqrt[3]{abc}$.
5. $2m\sqrt[4]{2m}$.
6. $3ab\sqrt[5]{abc}$.
7. $2mn\sqrt[3]{m^2n^2}$.
8. $a^2m^3\sqrt[3]{2am^2}$.
9. $(a+b)\sqrt{a+b}$.
10. $(x^2-y^2)\sqrt{x}$.
11. $x\sqrt{x^2-y^2}$.
12. $(p+q)\sqrt{p-q}$.
13. $(p-q)\sqrt{p+q}$.
14. $(p^2-q^2)\sqrt{p-q}$.
15. $(x+y)\sqrt[3]{x+y}$.

Simplify the following:

16. $2a\sqrt{\frac{3a}{25b^2}}$.
17. $3ab\sqrt{\frac{2ab}{(a+b)^4}}$.
18. $4p^2\sqrt[3]{\frac{5p^2}{343q^9}}$.
19. $3xy\sqrt[3]{\frac{x^2y^2}{(x+y)^6}}$.
20. $\sqrt{\frac{a^2b^4c^6}{(a+b+c)^2}}$.
21. $\sqrt[5]{\frac{32a^5b}{243x^{10}y^{15}}}$.

143. The *m*th Root of the *n*th Root. If we separate a number into two equal factors, and each of these factors into three equal factors, we thereby separate the number into six equal factors.

For example, $64 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. If we separate 64 into two equal factors, each factor equals $2 \cdot 2 \cdot 2$, or 8; and if we separate each of these into three equal factors, we have 2 for each factor, and this is one of the six equal factors of 64.

That is, $\sqrt[3]{\sqrt{64}} = \sqrt[3]{8} = 2$, and hence $\sqrt[3]{\sqrt{64}} = \sqrt[6]{64}$.

Therefore the cube root of the square root is the sixth root.

In the same way, *the mth root of the nth root of a number equals the mnth root of the number.*

That is, $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Thus $\sqrt{\sqrt[3]{729}} = \sqrt[6]{729}$, and $\sqrt[3]{\sqrt{729}} = \sqrt[6]{729}$.

Similarly, $\sqrt[mn]{a^{nx}} = \sqrt[m]{\sqrt[n]{a^{nx}}} = \sqrt[m]{a^x}$.

Thus $\sqrt[6]{49} = \sqrt[3]{\sqrt{49}} = \sqrt[3]{7}$; $\sqrt[10]{32} = \sqrt{\sqrt[5]{32}} = \sqrt{2}$.

Exercise 67. Radicals

Simplify the following:

1. $\sqrt[4]{36}$.
2. $\sqrt[10]{1024}$.
3. $\sqrt[6]{125}$.
4. $\sqrt[4]{100}$.
5. $\sqrt[6]{343}$.
6. $\sqrt[6]{4\,a^2b^2}$.
7. $\sqrt[4]{25\,a^4b^2}$.
8. $\sqrt[8]{36\,a^4b^4}$.
9. $\sqrt[6]{343\,a^3x^6}$.
10. $\sqrt[8]{121\,m^2n^2}$.
11. $\sqrt[9]{a^3b^6}$.
12. $\sqrt[10]{a^5b^{10}c^{15}}$.
13. $\sqrt[4]{\frac{1}{16}\,a^2b^4c^6}$.
14. $\sqrt[6]{\frac{1}{64}\,a^3b^3c^3}$.
15. $\sqrt[3]{\frac{1}{216}\,a^{4n}b^{4n}}$.

Express as a single root:

16. $\sqrt{\sqrt[3]{a-b}}$.
17. $\sqrt[3]{\sqrt{a-b}}$.
18. $\sqrt{\sqrt[5]{a^2+1}}$.
19. $\sqrt{\sqrt{abc}}$.
20. $\sqrt{\sqrt[4]{abc}}$.
21. $\sqrt[m]{\sqrt{a-b^2}}$.
22. $\sqrt[5]{\sqrt{1+x^2}}$.
23. $\sqrt{\sqrt[m]{abcd}}$.
24. $\sqrt[2n+1]{\sqrt{ax}}$.
25. $\sqrt[2n-1]{\sqrt[3]{mn}}$.
26. $\sqrt[a+b]{\sqrt[4]{a+b}}$.
27. $\sqrt[a+b]{\sqrt[a-b]{abc}}$.

144. Fractional Exponent. The positive integral exponent shows how many times the number which it affects is taken as a factor.

Thus $x^2 = xx$, the exponent showing that x is taken twice as a factor.

There are also fractional exponents, but these must have a different meaning from integral exponents, for we cannot take x half of a time as a factor any more than we can pick up a book half of a time.

Since we have certain laws of positive integral exponents (§ 106), it would be illogical and inconvenient if we should give to fractional exponents a meaning such that these laws would not also apply to them. Therefore

Such a meaning must be given to fractional exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \cdot a^n = a^{m+n}$,

we must have $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$.

But $\sqrt{a} \cdot \sqrt{a} = a$.

We therefore define $a^{\frac{1}{2}}$ to mean $\sqrt{a}$.

And since $(a^m)^n = a^{mn}$,

we must have $(a^{\frac{1}{2}})^3 = a^{\frac{3}{2}}$.

That is, we define $a^{\frac{3}{2}}$ to mean $(\sqrt{a})^3$.

Therefore we see that the following meaning must be given to a fractional exponent if the laws of exponents are to remain valid.

In a fractional exponent the numerator indicates the power of the number affected, and the denominator indicates the root.

Thus $4^{\frac{1}{2}}$ means the same as $\sqrt{4}$ and equals 2;

$8^{\frac{2}{3}}$ means the same as $(\sqrt[3]{8})^2$, or as $\sqrt[3]{8^2}$ and equals 4,

and $a^{\frac{m}{n}}$ means the same as $(\sqrt[n]{a})^m$, or as $\sqrt[n]{a^m}$.

Sometimes it is much easier to use fractional exponents than radicals. In simple cases, like $\sqrt{2}$ or $\sqrt[3]{8}$, it is easier to write the radical sign.

145. Comparison of Fractional Exponents and Radicals. The following comparison of fractional exponents with the ordinary radical sign should be studied with care:

Fractional Exponent	*Ordinary Radical Sign*	
$(a^{\frac{1}{n}})^n = a.$	$(\sqrt[n]{a})^n = a.$	§ 133
$(a^n)^{\frac{1}{n}} = a.$	$\sqrt[n]{a^n} = a.$	§ 135
$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}.$	$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$	§ 134
$(a^{mn})^{\frac{1}{n}} = a^m.$	$\sqrt[n]{a^{mn}} = a^m.$	§ 135
$ab^{\frac{1}{n}} = (a^n b)^{\frac{1}{n}}.$	$a\sqrt[n]{b} = \sqrt[n]{a^n b}.$	§ 141
$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$	§ 142
$(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}.$	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$	§ 143

146. Reduction of Fractional Exponents. Since the nth root of any power of a number is found by dividing the exponent of the power by the index of the root (§ 135), in the expression $a^{\frac{m}{n}}$ we may divide m by n just as if the exponent $\frac{m}{n}$ were an ordinary fraction. That is,

A fractional exponent may be reduced in the same way as an ordinary fraction.

For example, $a^{\frac{2}{2}} = a^1$, $x^{\frac{3}{3}} = x^{\frac{4}{4}}$, $m^{\frac{4}{4}} = m$, $(a+b)^{\frac{5}{5}} = (a+b)^{\frac{1}{1}}$, and $(a-b)^{\frac{6}{2}} = (a-b)^3$.

147. Advantages of Fractional Exponents. If we have a complicated expression involving radicals, it is usually easier to employ fractional exponents.

For example, $\left(\sqrt[4]{\sqrt[3]{a^2}}\right)^3 = (a^{\frac{2}{3}})^{\frac{3}{4}} = a^{\frac{2}{3} \cdot \frac{3}{4}} = a^{\frac{1}{2}}$.

Similarly, $\left(\sqrt[m]{\sqrt[n]{a^p}}\right)^q = \left(a^{\frac{p}{n}}\right)^{\frac{q}{m}} = a^{\frac{pq}{mn}}$.

Exercise 68. Fractional Exponents

Express the following with radical signs:

1. $x^{\frac{1}{2}}$.
2. $x^{\frac{2}{3}}$.
3. $x^{\frac{7}{4}}$.
4. $ax^{\frac{7}{4}}$.
5. $(ax)^{\frac{7}{4}}$.
6. $15\,m^{\frac{1}{2}}$.
7. $15\,m^{\frac{2}{3}}$.
8. $15^{\frac{5}{6}}m^{\frac{5}{6}}$.
9. $(15\,m)^{\frac{5}{6}}$.
10. $(25\,ab)^{\frac{7}{16}}$.
11. $2\,a^{\frac{1}{2}}b^{\frac{1}{2}}$.
12. $2\,a^{\frac{1}{2}}b^{\frac{2}{3}}$.
13. $2\,a^{\frac{2}{3}}b^{\frac{2}{3}}$.
14. $2^{\frac{2}{3}}a^{\frac{2}{3}}b^{\frac{2}{3}}$.
15. $(2\,ab)^{\frac{2}{3}}$.
16. $x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}$.
17. $(xyz)^{\frac{2}{3}}$.
18. $a^{1\frac{5}{12}}b^{\frac{5}{6}}c^{\frac{5}{6}}$.
19. $a^{1\frac{5}{12}}b^{1\frac{7}{12}}c^{1\frac{1}{12}}$.
20. $a^{\frac{1}{x}}b^{\frac{1}{y}}c^{\frac{1}{z}}$.

Simplify the following expressions:

21. $121^{\frac{1}{2}}$.
22. $625^{\frac{1}{2}}$.
23. $625^{\frac{1}{4}}$.
24. $625^{\frac{3}{4}}$.
25. $729^{\frac{1}{3}}$.
26. $729^{\frac{2}{3}}$.
27. $729^{\frac{1}{6}}$.
28. $729^{\frac{5}{6}}$.
29. $243^{\frac{1}{5}}$.
30. $243^{\frac{4}{5}}$.
31. $216^{\frac{1}{3}}$.
32. $-216^{\frac{1}{3}}$.
33. $(-216)^{\frac{1}{3}}$.
34. $-216^{\frac{2}{3}}$.
35. $(-216)^{\frac{2}{3}}$.
36. $(-512)^{\frac{1}{3}}$.
37. $(-512)^{\frac{2}{3}}$.
38. $(-512)^{\frac{1}{9}}$.
39. $(-512)^{\frac{2}{9}}$.
40. $(-512)^{\frac{5}{9}}$.
41. $64^{\frac{2}{3}} \cdot 343^{\frac{1}{3}}$.
42. $64^{\frac{1}{3}} \cdot 343^{\frac{2}{3}}$.
43. $(\frac{1}{81})^{\frac{1}{2}} \cdot (\frac{1}{81})^{\frac{3}{4}}$.
44. $25^{\frac{1}{2}} \cdot (\frac{1}{8})^{\frac{1}{3}} \cdot 16^{\frac{3}{4}}$.
45. $625^{\frac{3}{4}} \cdot 1331^{\frac{1}{3}}$.
46. $144^{\frac{1}{2}} \cdot (-1331)^{\frac{2}{3}}$.
47. $(-1728)^{\frac{2}{3}} \cdot (-1331)^{\frac{2}{3}}$.
48. $2197^{\frac{1}{3}} \cdot 3375^{\frac{1}{3}}$.
49. $(-2197)^{\frac{1}{3}} \cdot 3375^{\frac{2}{3}}$.
50. $(-2197)^{\frac{2}{3}} \cdot (-3375)^{\frac{2}{3}}$.

Express with fractional exponents:

51. $\sqrt[3]{ab}$.
52. $\sqrt[3]{a^2b^2}$.
53. $\sqrt[3]{ab^2}$.
54. $\sqrt[4]{ab^3c^3}$.
55. $\sqrt[5]{a^2b^3c^2}$.
56. $\sqrt[5]{a^2b^3c^4}$.
57. $\sqrt[5]{32\,a^5b^{10}c^{15}}$.
58. $2\sqrt{2}\sqrt[3]{2}$.
59. $2\sqrt{a}\sqrt[3]{b^2}$.
60. $a\sqrt{x}\sqrt[3]{x^2y^2}$.
61. $p\sqrt[3]{q^2}\sqrt[4]{q^3}$.
62. $a^2b^2\sqrt[3]{a^2b^2}\sqrt[4]{a^3b^3}$.
63. $xyz\sqrt[5]{x^2y^3z^4}$.
64. $3\,ab^2c^3\sqrt[5]{ab^2c^3}$.
65. $\sqrt[5]{ab}\sqrt[6]{bc}$.
66. $\sqrt[5]{a^2b^4}\sqrt[6]{b^5c^3}$.
67. $\sqrt[3]{-a^2b^2c^2d^2}$.
68. $\sqrt[3]{a^2+b^2+c^2}$.
69. $\sqrt[5]{a^3-b^3+c^3}$.
70. $\sqrt[7]{a^7b^7c^7d^7}$.
71. $\sqrt[7]{a^7+b^7+c^7}$.

148. Reduction of Surds to a Common Index. From the meaning of the fractional exponent (§ 144) and the fact that such an exponent may be reduced like an ordinary fraction (§ 146), we know that

$$\sqrt[5]{a^4} = a^{\frac{4}{5}} = a^{\frac{12}{15}} = \sqrt[15]{a^{12}},$$

and

$$\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{\frac{10}{15}} = \sqrt[15]{a^{10}}.$$

In the same way we may reduce any two surds.

That is,

$$\sqrt[m]{a^k} = a^{\frac{k}{m}} = a^{\frac{kn}{mn}} = \sqrt[mn]{a^{kn}}.$$

149. Similar Surds. Surds which, when reduced to their simplest forms, have the same surd factor are called *similar surds.*

For example, $3\sqrt{2}$, $4\sqrt{2}$, and $9\sqrt{2}$ are similar surds.

150. Addition and Subtraction of Surds. The principles of addition and subtraction of similar surds are the same as those of other algebraic expressions.

Thus

$$3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}.$$

Exercise 69. Reduction, Addition, and Subtraction

Reduce to surds having a common root index:

1. $\sqrt{7}$, $\sqrt[3]{3}$.
2. $\sqrt[3]{5}$, $\sqrt[4]{3}$.
3. $\sqrt[4]{7}$, $\sqrt[5]{2}$.
4. $m^{\frac{1}{4}}$, $n^{\frac{2}{3}}$.
5. $x^{\frac{2}{3}}$, $y^{\frac{1}{5}}$.
6. $a^{\frac{1}{7}}$, $b^{\frac{2}{3}}$.
7. $\sqrt{m}$, $m^{\frac{1}{4}}$.
8. $3\sqrt{2}$, $2\sqrt[3]{3}$.
9. $a^{\frac{1}{2}}$, $b^{\frac{1}{3}}$, $c^{\frac{3}{4}}$.

Express with similar surds and simplify:

10. $15\sqrt{2} + 7\sqrt{2} + 5\sqrt{8} + 3\sqrt{18} - 2\sqrt{32}$.
11. $5\sqrt[3]{5} + 4\sqrt[3]{40} - 2\sqrt[3]{135} + \sqrt[3]{5000}$.
12. $7\sqrt{x} - 3\sqrt{y} + 2\sqrt{x} - 4\sqrt{y} + \sqrt{4y} - \sqrt{9x}$.
13. $28^{\frac{1}{2}} + 63^{\frac{1}{2}} + 112^{\frac{1}{2}} - \sqrt{11} + 2\sqrt{11} - 5\sqrt{11} + 2 - \sqrt{4}$.
14. $\sqrt{(a+b-c)^2 d} + \sqrt{(a-b+c)^2 d} - 2a\sqrt{d} + d^{\frac{3}{2}} - \sqrt[3]{d^2}$.
15. $2\sqrt[3]{54} - 7\sqrt[3]{16} + \sqrt[3]{2662} - 2\sqrt[3]{128} + \sqrt[3]{-250}$.

151. Multiplication of Surds. Since we have found (§ 134) that $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$, therefore

To multiply one surd by another, express the surds with a common index. Then find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor.

In all work with radicals the results should be reduced to their simplest forms.

152. Multiplication of Polynomial Radicals. Polynomial radicals are multiplied like ordinary polynomials. For example,

$$\begin{array}{r} 3\sqrt{5} + 2\sqrt{3} \\ 7\sqrt{5} - 4\sqrt{3} \\ \hline 105 + 14\sqrt{15} \\ \quad - 12\sqrt{15} - 24 \\ \hline 105 + 2\sqrt{15} - 24 \end{array} = 81 + 2\sqrt{15}.$$

Exercise 70. Multiplication of Radicals

Multiply the following:

1. $(\sqrt{3} + 5\sqrt{12} + 6\sqrt{27} + 15\sqrt{48})\sqrt{3}$.
2. $(\sqrt{7} + 9\sqrt{28} - 2\sqrt{63} + 11\sqrt{112} - 6\sqrt{175})\sqrt{7}$.
3. $(\sqrt[3]{5} + 12\sqrt[3]{40} - 2\sqrt[3]{135} + 6\sqrt[3]{320} - 5\sqrt[3]{625})\sqrt[3]{5}$.
4. $(\sqrt[3]{2} + 7\sqrt[3]{16} - 5\sqrt[3]{54} + 9\sqrt[3]{250} + 6\sqrt[3]{432}) \cdot 6\sqrt[3]{2}$.
5. $(\sqrt{20} + 6\sqrt{45} + \sqrt{720} - 3\sqrt{80} + 5\sqrt{500} - 3\sqrt{180}) \cdot 7\sqrt{5}$.
6. $(2+\sqrt{7})(2-\sqrt{7})$; $(2-\sqrt{7})(2-\sqrt{7})$; $(\sqrt{7}-2)(\sqrt{7}-2)$.
7. $(3\sqrt{5} + 2\sqrt{7})(3\sqrt{5} - 2\sqrt{7})$; $(\sqrt{5} - \sqrt{7})(\sqrt{5} - \sqrt{7})$.
8. $(6 + 2\sqrt{5} + 5\sqrt{2})(6 + 2\sqrt{5} - 5\sqrt{2})$.
9. $(2\sqrt{7} + 7\sqrt{2})(2\sqrt{7} - 7\sqrt{2})$.
10. $(\sqrt[3]{9} - 8\sqrt[3]{72} + 9\sqrt[3]{1125})(5\sqrt[3]{2} + 3\sqrt[3]{3} - 6\sqrt[3]{9})$.
11. $(\sqrt{x} + \sqrt{y})(x^2 + xy + y^2)(\sqrt{x} - \sqrt{y})$.

153. Division of Radicals. If, in the solution of a problem, we find $\frac{3}{\sqrt{5}}$ as the result, we have an awkward numerical expression. If we extract the square root of 5 to several decimal places, we have a long divisor. But if we multiply both terms by $\sqrt{5}$, we have $\frac{3\sqrt{5}}{5}$, and the division is much simpler.

Comparing the two operations we have

$$(1)\quad \frac{3}{\sqrt{5}} = \frac{3}{2.236} = 1.3416, \text{ nearly.}$$

$$(2)\quad \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5} = \frac{6.708}{5} = 1.3416, \text{ nearly.}$$

The work in (1) will be found more difficult than that in (2).

Therefore, *if the denominator of a fraction contains a surd, multiply both terms of the fraction by a number that will make the denominator rational.*

This is called *rationalizing the denominator*, the expression meaning that the denominator is to be made rational without changing the value of the fraction.

154. Division by a Binomial Surd. If a divisor or the denominator of a fraction is a binomial involving only quadratic surds, it may be rationalized by the process shown in the following example:

Simplify $\dfrac{2+3\sqrt{2}}{3-5\sqrt{2}}$.

Multiply both terms of the fraction by $3+5\sqrt{2}$, because the denominator will then become the difference of two squares. Then

$$\frac{2+3\sqrt{2}}{3-5\sqrt{2}} = \frac{(3+5\sqrt{2})(2+3\sqrt{2})}{(3+5\sqrt{2})(3-5\sqrt{2})} = \frac{36+19\sqrt{2}}{9-50} = -\frac{1}{41}(36+19\sqrt{2}).$$

The binomial $3+5\sqrt{2}$ is called the *conjugate* of $3-5\sqrt{2}$. In such cases we multiply both terms of the fraction by the conjugate of the denominator, thus rationalizing the denominator.

In case the denominator has more than two terms we may proceed as suggested in Ex. 23, page 152.

Exercise 71. Division by Surds

Divide as indicated:

1. $\sqrt{7} \div \sqrt{2}$.
2. $\sqrt[3]{7} \div \sqrt{2}$.
3. $2\sqrt[3]{5} \div \sqrt[3]{2}$.
4. $-3\sqrt{5} \div \sqrt[3]{2}$.
5. $\sqrt{7} \div \sqrt[4]{7}$.
6. $\sqrt[4]{7} \div \sqrt{7}$.
7. Divide $\sqrt{5}$ by $\sqrt{2}+\sqrt{3}$, and $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}$.

Rationalize the denominators:

8. $\dfrac{7+\sqrt{3}}{7-\sqrt{3}}$.
9. $\dfrac{3+\sqrt{6}}{3-\sqrt{6}}$.
10. $\dfrac{4-5\sqrt{2}}{4+5\sqrt{2}}$.
11. $\dfrac{9-3\sqrt{3}}{9+3\sqrt{3}}$.
12. $\dfrac{2+\sqrt{5}}{3-\sqrt{5}}$.
13. $\dfrac{9+\sqrt{7}}{8-\sqrt{7}}$.
14. $\dfrac{5-\sqrt{7}}{5+\sqrt{5}}$.
15. $\dfrac{5-\sqrt{5}}{5+\sqrt{5}}$.
16. $\dfrac{2a+\sqrt{b}}{2a-\sqrt{b}}$.
17. $\dfrac{2a-\sqrt{b}}{2a+\sqrt{b}}$.
18. $\dfrac{a+c\sqrt{b}}{c-a\sqrt{d}}$.
19. $\dfrac{x^2+y\sqrt{y}}{x^2-y\sqrt{y}}$.

20. Simplify $\dfrac{9+\sqrt{7}}{9-\sqrt{7}}$, and find the approximate value to three decimal places.

21. Simplify $\dfrac{5+2\sqrt{5}}{5-2\sqrt{5}}$, and find the approximate value to three decimal places.

22. Arrange in order of magnitude, beginning with the largest: $\dfrac{4+\sqrt{7}}{4-\sqrt{7}}$, $\dfrac{4+\sqrt{5}}{4-\sqrt{5}}$, and $\dfrac{5+\sqrt{6}}{5-\sqrt{6}}$.

23. In the fraction $\dfrac{2+\sqrt{3}+\sqrt{7}}{2+\sqrt{3}-\sqrt{7}}$, multiply both terms by $2+\sqrt{3}+\sqrt{7}$. Then rationalize the resulting denominator.

24. As in Ex. 23, rationalize the denominator of the fraction

$$\frac{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}-1}{3.5\sqrt{2}+\frac{1}{2}\sqrt{6}+1}.$$

155. Powers of Radicals. The powers of radicals are conveniently found by the following methods:

1. Find the square of $7\sqrt[3]{5^2}$ and the cube of $5\sqrt{x}$.

$$(7\sqrt[3]{5^2})^2 = (7 \cdot 5^{\frac{2}{3}})^2 = 7^2 \cdot 5^{\frac{4}{3}} = 49 \cdot 5 \cdot 5^{\frac{1}{3}} = 245\sqrt[3]{5}.$$

$$(5\sqrt{x})^3 = (5x^{\frac{1}{2}})^3 = 125x^{\frac{3}{2}} = 125xx^{\frac{1}{2}} = 125x\sqrt{x}.$$

2. Find the square of $5 + \sqrt{3}$.

$$(5 + \sqrt{3})^2 = 25 + 10\sqrt{3} + 3$$
$$= 28 + 10\sqrt{3}.$$

Exercise 72. Powers of Radicals

Raise the following to the second power:

1. $\sqrt{2}$.
2. $\sqrt[3]{2}$.
3. $\sqrt[3]{4}$.
4. $\sqrt{a}$.
5. $\sqrt[3]{a^2}$.
6. $\sqrt[6]{m}$.
7. $\sqrt[6]{m^3}$.
8. $\sqrt[6]{m^5}$.
9. $(abc)^{\frac{1}{2}}$.
10. $(a-b)^{\frac{1}{2}}$.
11. $(a+x)^{\frac{1}{2}}$.
12. $(a^2+b^2)^{\frac{1}{2}}$.

Perform the operations indicated:

13. $(\sqrt{a})^3$.
14. $(\sqrt[3]{a})^2$.
15. $(\sqrt[4]{a})^2$.
16. $(\sqrt[4]{a^3})^2$.
17. $(a^{\frac{1}{2}}b^{\frac{1}{2}})^4$.
18. $(\sqrt{ab})^4$.
19. $(\sqrt[3]{-x^3y^3})^2$.
20. $(\sqrt{75})^3$.
21. $(-\sqrt[4]{16})^2$.
22. $(x^{\frac{1}{2}}y^{\frac{1}{4}})^8$.
23. $(x^{\frac{1}{3}}y^{\frac{1}{4}})^2$.
24. $(\sqrt[6]{ab})^3$.
25. $(\sqrt[3]{ab})^6$.
26. $(\sqrt[4]{ab})^6$.
27. $(\sqrt[6]{ab})^4$.
28. $(a^{\frac{1}{3}}b^{\frac{1}{6}})^4$.
29. $\sqrt[3]{\sqrt{8}}$.
30. $\sqrt[3]{\sqrt{64}}$.
31. $\sqrt{\sqrt[3]{64}}$.
32. $\sqrt[3]{\sqrt{x^3}}$.
33. $\sqrt[4]{\sqrt{x^8y^{16}}}$.
34. $\sqrt[3]{\sqrt[4]{x^{12}y^{24}}}$.
35. $[(a-b)^{\frac{1}{3}}]^6$.
36. $\sqrt[3]{(a-b)^6}$.
37. $[(a-b)^{\frac{1}{3}}]^{\frac{1}{3}}$.
38. $(\sqrt{a+x})^4$.
39. $(\sqrt[4]{a^2+x^2})^2$.
40. $[(1-x^2)^2]^{\frac{1}{2}}$.
41. $(\sqrt[3]{(a+b)^6})^{\frac{1}{2}}$.
42. $\sqrt[4]{(a+b+c)^2}$.
43. $[(\sqrt{a+b+c})^{\frac{1}{2}}]^4$.
44. $(\sqrt[3]{a^2-2ab+b^2})^{\frac{3}{2}}$.
45. $(\sqrt[5]{a^2+4a+4})^{10}$.

156. The Square Root of a Binomial Surd. Since $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b$, if we can write the binomial surd $x + \sqrt{y}$ in the form $a + 2\sqrt{ab} + b$, we can find the square root by inspection.

For example, find the square root of $7 + 4\sqrt{3}$.

We may write $7 + 4\sqrt{3} = 7 + 2\sqrt{12}$

$$= 4 + 2\sqrt{4 \cdot 3} + 3 = (2 + \sqrt{3})^2.$$

Therefore $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$.

To find the square root of a binomial surd, write it in the form $a + 2\sqrt{ab} + b$. *Then the square root will be in the form* $\sqrt{a} + \sqrt{b}$.

That is, $\sqrt{a + 2\sqrt{ab} + b} = \sqrt{a} + \sqrt{b}$.

Exercise 73. Square Root of a Binomial Surd

Examples 1 to 3, oral — Examples 4 to 27, written

1. Square $2 - \sqrt{3}$; $\sqrt{2} + \sqrt{3}$; $\sqrt{2} + 3$; $\sqrt{2} - \sqrt{3}$.
2. Square $2\sqrt{3}$; $2 + 2\sqrt{3}$; $3 + 2\sqrt{3}$; $3 - 2\sqrt{3}$.
3. Square $a + \sqrt{b}$; $a + 2\sqrt{b}$; $\sqrt{a} - \sqrt{b}$; $\sqrt{a} - 2\sqrt{b}$.

Find the square roots of the following expressions:

4. $5 + 2\sqrt{6}$.
5. $11 + 6\sqrt{2}$.
6. $21 - 12\sqrt{3}$.
7. $17 + 12\sqrt{2}$.
8. $9 - 4\sqrt{5}$.
9. $6 - 2\sqrt{5}$.
10. $8 + 2\sqrt{15}$.
11. $36 + 18\sqrt{3}$.
12. $30 + 12\sqrt{6}$.
13. $23 - 4\sqrt{15}$.
14. $9 + 2\sqrt{14}$.
15. $12 + 2\sqrt{35}$.

Solve the following equations:

16. $x^2 = 12 + 2\sqrt{11}$.
17. $x^2 = 18 + 8\sqrt{2}$.
18. $x^2 = 27 + 10\sqrt{2}$.
19. $x^2 = 51 - 14\sqrt{2}$.
20. $x^2 = 66 - 16\sqrt{2}$.
21. $2(x^2 - 2) = \sqrt{15}$.
22. $x^2 - 16 = 2\sqrt{55}$.
23. $x^2 - 11 = 4\sqrt{7}$.
24. $x^2 - 23 = 8\sqrt{7}$.
25. $x^2 - 32 = 10\sqrt{7}$.
26. $x^2 - 15 = 4\sqrt{11}$.
27. $x^2 + 14\sqrt{11} = 60$.

157. Negative Exponents. We have found the meaning of positive integral exponents as follows:

a^3 means aaa, a^n means $aaa \cdots$ (n factors).

We have also found the meaning of fractional exponents as follows:

$a^{\frac{1}{3}}$ means $\sqrt[3]{a^1}$ or $(\sqrt[3]{a})^1$, $a^{\frac{m}{n}}$ means $\sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$.

We also have occasion to use negative exponents in algebra, and their meaning will now be considered. As with fractional exponents,

Such a meaning must be given to negative exponents as will make the laws of exponents valid for them as well as for positive integral exponents.

Since $a^m \div a^n = a^{m-n}$, we know that $a^5 \div a^2 = a^3$. We must therefore have $a^2 \div a^5 = a^{2-5} = a^{-3}$. But $a^2 \div a^5 = \frac{1}{a^3}$. Hence a^{-3} must equal $\frac{1}{a^3}$.

More generally, we must have $a^m \cdot a^{-n} = a^{m-n}$; but $a^m \cdot \frac{1}{a^n} = a^{m-n}$, and hence a^{-n} must equal $\frac{1}{a^n}$.

Therefore, *a quantity affected by a negative exponent equals the reciprocal of that quantity affected by a numerically equal positive exponent.*

That is, $$a^{-n} = \frac{1}{a^n}.$$

For example, $a^{-2} = \frac{1}{a^2}$, $a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}$,

$a^{-3} = \frac{1}{a^3}$, $a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}$.

If we have a complicated expression involving radicals and fractions it is usually easier to employ negative and fractional exponents than to use positive exponents and radical signs.

For example, $\sqrt[5]{\frac{\sqrt{a^3}}{\sqrt[3]{a^2}}} = \left(a^{\frac{3}{2}} \cdot a^{-\frac{2}{3}}\right)^{\frac{1}{5}} = \left(a^{\frac{3}{2}-\frac{2}{3}}\right)^{\frac{1}{5}} = \left(a^{\frac{5}{6}}\right)^{\frac{1}{5}} = a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

158. Zero as an Exponent. It now remains to find the meaning of zero as an exponent. If a^0 is to conform to the same laws as a^m, we must have $a^0 \cdot a^m = a^{0+m} = a^m$. But to have this true, a^0 must equal 1. Therefore

A quantity affected by an exponent zero is equal to 1.

That is, $$a^0 = 1.$$

For example, $2^0 = 1$, $(\frac{1}{2})^0 = 1$, $(-1)^0 = 1$.

Exercise 74. Negative and Zero Exponents

Examples 1 to 22, oral — Examples 23 to 31, written

1. Express a^{-5}, using a positive exponent.

2. What is the value of 7^0? of $(-3)^0$? of $(\frac{1}{4})^0$? of m^0? of 4^0? of 4^{-1}? of $4^{\frac{1}{2}}$? of $4^{\frac{3}{2}}$?

Express with positive exponents:

3. a^{-7}.	**6.** $(\frac{3}{4})^{-1}$.	**9.** $8^{-\frac{1}{3}}$.	**12.** $a^{-\frac{3}{5}}$.
4. x^{-9}.	**7.** $(\frac{1}{2})^{-2}$.	**10.** $8^{-\frac{2}{3}}$.	**13.** $(a+b)^{-1}$.
5. $(\frac{1}{3})^{-1}$.	**8.** $(\frac{1}{3})^{-3}$.	**11.** $27^{-\frac{2}{3}}$.	**14.** $a^{-1}+b^{-1}$.

Express in integral form, using negative and fractional exponents:

15. $\frac{1}{2}$.	**17.** $\frac{3}{5}$.	**19.** $\frac{1}{2^5}$.	**21.** $\frac{1}{81}$.
16. $\dfrac{1}{a}$.	**18.** $\dfrac{2}{a^2}$.	**20.** $\dfrac{a}{\sqrt{b}}$.	**22.** $\dfrac{a}{b\sqrt{c}}$.

Copy and express in integral form, using negative and fractional exponents:

23. $\dfrac{ab}{xy}$.	**26.** $\dfrac{a+b}{a-b}$.	**29.** $\dfrac{1}{a^2+2ab+b^2}$.
24. $\dfrac{ab}{\sqrt{xy}}$.	**27.** $\dfrac{a+b}{\sqrt{a-b}}$.	**30.** $\dfrac{1}{\sqrt[3]{a^2-2ab+b^2}}$.
25. $\dfrac{pq}{\sqrt[3]{a^2b^2}}$.	**28.** $\dfrac{a-b}{\sqrt[3]{(a+b)^2}}$.	**31.** $\dfrac{1}{a^2+b^2}(a^2-b^2)$.

159. Review of the Meaning of a^0 and a^{-n}. As already explained (§ 158), $a^0 = 1$, and (§ 157) $a^{-n} = \frac{1}{a^n}$. The reasonableness of this is further seen from the following relations:

Since	a^3	means	aaa,	or	$a^4 \div a$,
and	a^2	means	aa,	or	$a^3 \div a$,
it should follow that	a^1	means	a,	or	$a^2 \div a$,
and that	a^0	means	1,	or	$a \div a$,
and that	a^{-1}	means	$\frac{1}{a}$,	or	$1 \div a$,
and so on, so that	a^{-n}	means	$\frac{1}{a^n}$.		

160. Review of the Meaning of $a^{\frac{1}{n}}$. As already explained (§ 144), $a^{\frac{1}{n}} = \sqrt[n]{a}$. The reasonableness of this is further seen from the following relations:

Starting with a^4,

if this exponent is half as large, we have a^2, or $\sqrt{a^4}$;

if this exponent is half as large, we have a, or $\sqrt{a^2}$;

if this exponent is half as large, $a^{\frac{1}{2}}$ should equal $\sqrt{a}$;

and in the same way $a^{\frac{1}{n}}$ should equal $\sqrt[n]{a}$.

We thus see that not only do a^0, a^{-n}, $a^{\frac{m}{n}}$, have the meanings assigned to them because they obey the fundamental laws of exponents (§§ 158, 157, and 144), but because these meanings fit into the general definitions.

161. Need for Different Symbols. It may be asked why we need two different symbols for fractions and roots. That is, if $\frac{a}{b}$ signifies a fraction, why do we need ab^{-1}, and if $\sqrt{a}$ signifies the square root, why do we need $a^{\frac{1}{2}}$?

Sometimes it is more convenient to use 0.50 than $\frac{1}{2}$, and $33\frac{1}{3}\%$ instead of $\frac{1}{3}$. Similarly, it is sometimes more convenient, as shown in §§ 147 and 157, to use the fractional exponent instead of the radical sign, although the latter is the more easily written, and sometimes the negative exponent is more simple than the fraction.

162. Laws of Exponents. The following laws have already been proved for positive integral exponents (§ 106) and assumed for other kinds of exponents (§§ 144 and 157):

$$\text{Law I.} \quad a^m a^n = a^{m+n}.$$

$$\text{Law II.} \quad a^m \div a^n = a^{m-n}.$$

$$\text{Law III.} \quad (ab)^n = a^n b^n.$$

$$\text{Law IV.} \quad (a^m)^n = a^{mn}.$$

In any of these cases m or n, or both m and n, may be either fractional or negative, or both fractional and negative, as well as positive and integral. To prove all possible cases would take too much time, but a few typical ones will now be considered. All of the others admit of similar treatment.

It is unnecessary to master all of the proofs here given, and the teacher may safely omit one or more. The important thing is that the student should know that the fundamental laws hold for all kinds of exponents.

1. To prove that $a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$.

To prove this we must first recall that $\frac{m}{n}$ is here a symbol meaning that the nth root of the mth power is to be taken, and that it is not a fraction in the ordinary sense. We are now to show that it may be reduced like an ordinary fraction, and we are to show this by the laws of positive integral exponents already proved.

Let	$x = a^{\frac{m}{n}}$.	
Then	$x^n = a^m$,	Why?
and	$x^{pn} = a^{pm}$.	Why?
Extracting roots,	$x = a^{\frac{pm}{pn}}$.	
Therefore	$a^{\frac{m}{n}} = a^{\frac{pm}{pn}}$, since each equals x.	

2. To prove that $(ab)^m = a^m b^m$ when m is negative ($m = -n$), that is, that $(ab)^{-n} = a^{-n} b^{-n}$.

We have, by the definition of negative exponent and by a law already proved,

$$(ab)^{-n} = \frac{1}{(ab)^n} = \frac{1}{a^n b^n} = a^{-n} b^{-n}.$$

3. To prove that $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$.

Let $x = (ab)^{\frac{p}{q}}$.

Then $x^q = (ab)^p = a^p b^p$. Why?

$\therefore x = \sqrt[q]{a^p} \cdot \sqrt[q]{b^p} = a^{\frac{p}{q}} b^{\frac{p}{q}}$. Why?

$\therefore (ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$. Why?

4. To prove that $a^{-p} a^{-q} = a^{-p-q}$.

$$a^{-p} a^{-q} = \frac{1}{a^p} \cdot \frac{1}{a^q} = \frac{1}{a^{p+q}} = a^{-p-q}.$$ Explain

5. To prove that $a^{\frac{m}{n}} a^{\frac{p}{q}} = a^{\frac{mq+np}{nq}}$.

We have $a^{\frac{m}{n}} a^{\frac{p}{q}} = a^{\frac{mq}{nq}} a^{\frac{np}{nq}}$, as in **1**

$= (a^{mq} a^{np})^{\frac{1}{nq}}$, as in **3**

$= (a^{mq+np})^{\frac{1}{nq}}$ Why?

$= a^{\frac{mq+np}{nq}}$. Why?

6. To prove that $(a^{-m})^{-n} = a^{mn}$.

We have $(a^{-m})^{-n} = \left(\frac{1}{a^m}\right)^{-n} = (a^m)^n = a^{mn}$. Explain

7. To prove that $(a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$.

Let $x = (a^{\frac{m}{n}})^{\frac{p}{q}}$.

Then $x^q = (a^{\frac{m}{n}})^p$ Why?

$= a^{\frac{m}{n}} a^{\frac{m}{n}} \cdots p$ factors,

$= a^{\frac{m+m+\cdots p \text{ terms}}{n}}$, as in **5**.

$\therefore x^{nq} = a^{mp}$, Why?

and $x = a^{\frac{mp}{nq}}$.

$\therefore (a^{\frac{m}{n}})^{\frac{p}{q}} = a^{\frac{mp}{nq}}$, since each equals x.

163. Advantage of Negative and Fractional Exponents. The advantage of using a^{-n} instead of $\frac{1}{a^n}$, and $a^{\frac{1}{n}}$ instead of $\sqrt[n]{a^s}$, already mentioned in §§ 147 and 157, should be fully appreciated.

Thus it is easier to see that $(a^{-\frac{2}{3}})^{-\frac{1}{2}} = a$, than that

$$\frac{1}{\sqrt{\left(\frac{1}{\sqrt[3]{a^2}}\right)^3}} = a,$$

although the two mean the same thing.

Exercise 75. Negative and Fractional Exponents

Examples 1 to 5, oral — Examples 6 to 44, written

1. Simplify $(a^{-2})^{-3}$; $(a^{\frac{1}{2}})^2$; $(a^{\frac{2}{3}})^3$; $(m^{\frac{3}{4}})^{\frac{4}{3}}$.
2. Expand $(a^{-1}+b^{-1})^2$; $(a^{-2}-b^{-2})^2$; $(a^{-2}+a^2)^2$.
3. Multiply $a^{-1}+b^{-1}$ by $a^{-1}-b^{-1}$; by $2\,a^{-1}$.
4. Divide $a^{-2}-b^{-2}$ by $a^{-1}+b^{-1}$; by $a^{-1}-b^{-1}$.
5. Expand $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2$; $(a^{\frac{1}{2}}+b^{-1})^2$.

Perform the operations indicated:

6. $(x^{-2}+x^{-1}+1)(x^{-2}-x^{-1}+1)$.
7. $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$.
8. $(x^2+x^{-2})(x^2-x^{-2})$.
9. $(a^{-\frac{1}{2}}+b^{-\frac{1}{2}})(a^{-\frac{1}{2}}-b^{-\frac{1}{2}})$.
10. $(p^2q^2+q^{-2})(p^2q^2-q^{-2})$.
11. $(a^{\frac{p}{q}}+a^{\frac{m}{n}})^2$.
12. $(a^{-\frac{p}{q}}+a^{\frac{p}{q}})^3$.
13. $(a^{-\frac{p}{q}}+a^{\frac{p}{q}})(a^{-\frac{p}{q}}-a^{\frac{p}{q}})$.
14. Divide $x^{-10}-x^{-5}+1$ by $x^{-2}-x^{-1}+1$.
15. Divide $a^{-3}+2\,a^{-2}b^{-1}-3\,b^{-3}$ by $a^{-1}-b^{-1}$.
16. Express $\sqrt[3]{x^2}+2\sqrt[3]{xy}+\sqrt[3]{y^2}$ with fractional exponents and then extract the square root.
17. Express $\sqrt{\frac{1}{a^4}+2+\sqrt[3]{a^4}}$ with fractional exponents and then extract the square root.

Perform the following multiplications:

18. $x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$.

19. $x^{\frac{3}{4}} + x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1$ by $x^{\frac{1}{4}} + x^{\frac{1}{2}} + 1$.

20. $8x^{\frac{3}{4}} - 4xy^{\frac{1}{4}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{4}}$ by $2x^{\frac{1}{2}} - 3y^{\frac{1}{4}}$.

21. $x - 6a^{\frac{1}{2}}x^{\frac{3}{4}} + 12a^{\frac{3}{4}}x^{\frac{1}{2}} - 8a$ by $x^{\frac{3}{4}} - 4a^{\frac{1}{4}}x^{\frac{1}{2}} + 4a^{\frac{3}{4}}$.

22. $x^{-3} + x^{-2} + x^{-1} + 1$ by $x^{-2} + x^{-1} + 1$.

23. $x^{-\frac{3}{2}} + 3x^{-1} + 3x^{-\frac{1}{2}} + 1$ by $x^{-1} + 2x^{-\frac{1}{2}} + 1$.

24. $x^{-\frac{3}{2}} - 6a^{\frac{1}{2}}x^{-1} + 12ax^{-\frac{1}{2}} - 8a^{\frac{3}{2}}$ by $x^{-1} - 4a^{\frac{1}{2}}x^{-\frac{1}{2}} - 4a$.

Perform the following divisions:

25. $acx^2y^{-1} - bcyx^{-1}$ by $cx^{-\frac{1}{2}}y^{-\frac{1}{2}}$.

26. $a^2 - 3a^{\frac{3}{2}} + 6a^{\frac{1}{2}} - 7a + 6a^{\frac{1}{2}} - 3a^{\frac{1}{2}} + 1$ by $a^{\frac{3}{2}} - a^{\frac{1}{2}} + 1$.

27. $\frac{9}{16}a^2 - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{13}{36}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}$ by $\frac{3}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.

28. $x^{-\frac{7}{3}} - 3x^{-2} + x^{-\frac{5}{3}} - 4 + 12x^{-\frac{1}{3}} - 4x^{-\frac{2}{3}}$ by $x^{-\frac{5}{3}} - 4$.

Find the square root of the following expressions:

29. $a^{\frac{4}{3}}b^{-1} + 4ab^{-\frac{1}{2}} - 2a^{\frac{3}{2}} - 12a^{\frac{1}{2}}b^{\frac{1}{2}} + 9b$.

30. $60mn^{\frac{5}{4}} - 4m^{\frac{1}{4}}n^{\frac{3}{4}} - 48m^{\frac{3}{4}}n^{\frac{1}{4}} + 16m^2n^2 + 25m^{\frac{3}{2}}n$.

31. $9x^{-\frac{4}{3}} + 24x^{-1}y^{-\frac{1}{3}} + 46x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40x^{-\frac{1}{3}}y^{-1} + 25y^{-\frac{4}{3}}$.

32. $a^{-4} + b^{-6} + c^{-8} + 2a^{-2}b^{-3} + 2a^{-2}c^{-4} + 2b^{-3}c^{-4}$.

Rationalize the denominators of the following fractions:

33. $\dfrac{3}{5^{\frac{1}{2}} - 2^{\frac{1}{2}}}$.

34. $\dfrac{3}{10^{\frac{1}{2}} - 6^{\frac{1}{2}}}$.

35. $\dfrac{6}{12^{\frac{1}{2}} + 5^{\frac{1}{2}}}$.

36. $\dfrac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}} - b^{\frac{1}{2}}}$.

37. $\dfrac{\sqrt{11} + \sqrt{5}}{\sqrt{11} - \sqrt{5}}$.

38. $\dfrac{1 + 4\sqrt{5}}{6 + 2\sqrt{3}}$.

39. $\dfrac{\sqrt{a} - \sqrt{3b}}{\sqrt{a} + \sqrt{3b}}$.

40. $\dfrac{\sqrt{2x} + \sqrt{7}}{\sqrt{2x} - \sqrt{7}}$.

41. $\dfrac{30}{2 - \sqrt{3} + \sqrt{5}}$.

42. $\dfrac{12}{3 + 2^{\frac{1}{2}} - 3^{\frac{1}{2}}}$.

43. $\dfrac{a}{b + (a + b)^{\frac{1}{2}}}$.

44. $\dfrac{x + \sqrt{x - y}}{x - \sqrt{x - y}}$.

164. Operations involving Negative, Zero, and Fractional Exponents. Since we have defined negative, zero, and fractional exponents so that they conform to the ordinary laws of exponents, we operate with them in the same way as with positive integral exponents.

In the following operations the quantities having negative exponents are compared with the fractional forms to show their relation. In solving the problems in Exercise 76, however, use negative exponents whenever they appear, without changing to fractional forms.

1. Multiply $x^{-1} + 2$ by $x^{-1} - 7$.

$$(x^{-1} + 2)(x^{-1} - 7)$$
$$= x^{-2} + 2x^{-1} - 7x^{-1} - 14$$
$$= x^{-2} - 5x^{-1} - 14.$$

$$\left(\frac{1}{x} + 2\right)\left(\frac{1}{x} - 7\right)$$
$$= \frac{1}{x^2} + \frac{2}{x} - \frac{7}{x} - 14$$
$$= \frac{1}{x^2} - \frac{5}{x} - 14.$$

It is seen that the quantities with the negative exponents occupy less space and are quite as easily written.

2. Divide $x^{-3} + 1$ by $x^{-1} + 1$.

$$\begin{array}{l|l} x^{-3} + 1 & x^{-1} + 1 \\ \hline x^{-3} + x^{-2} & x^{-2} - x^{-1} + 1 \\ \quad - x^{-2} + 1 & \\ \quad - x^{-2} - x^{-1} & \\ \hline \qquad\quad x^{-1} + 1 & \\ \qquad\quad x^{-1} + 1 & \\ \hline \end{array}$$

$$\begin{array}{l|l} \frac{1}{x^3} + 1 & \frac{1}{x} + 1 \\ \hline \frac{1}{x^3} + \frac{1}{x^2} & \frac{1}{x^2} - \frac{1}{x} + 1 \\ \quad - \frac{1}{x^2} + 1 & \\ \quad - \frac{1}{x^2} - \frac{1}{x} & \\ \hline \qquad\quad \frac{1}{x} + 1 & \\ \qquad\quad \frac{1}{x} + 1 & \\ \hline \end{array}$$

This work is more compact in form than the other and is more easily written. After a little experience it is as easily understood as the other.

3. Factor $x^{-2} - x^{-1} - 56$.

Since the two numbers whose product is -56 and whose sum is -1 are -8 and $+7$, therefore

$$x^{-2} - x^{-1} - 56 = (x^{-1} - 8)(x^{-1} + 7).$$

This is only another way of writing

$$\frac{1}{x^2} - \frac{1}{x} - 56 = \left(\frac{1}{x} - 8\right)\left(\frac{1}{x} + 7\right).$$

It is therefore an extension of ordinary factoring to include fractions.

Exercise 76. Negative Exponents

Examples 1 to 6, oral — Examples 7 to 33, written

1. What is the value of $4^{\frac{1}{2}}$? of 4^{-2}? of 4^{0}?
2. What is the value of $9^{\frac{1}{2}}$? of $9^{-\frac{1}{2}}$? of 9^{0}?
3. What is the value of $8^{\frac{1}{3}}$? of $8^{\frac{2}{3}}$? of $8^{-\frac{2}{3}}$? of 8^{0}?
4. What is the value of $16^{\frac{1}{4}}$? of $16^{\frac{3}{4}}$? of $16^{-\frac{3}{4}}$? of 16^{0}?
5. What is the value of 1^{-1}? of 1^{-2}? of 1^{-3}? of $1^{\frac{1}{2}}$? of 1^{0}?
6. What is the value of 2^{-1}? of 2^{-2}? of 2^{-3}? of 2^{0}?

Multiply:

7. $(x^{-1}+7)(x^{-2}-3)$.
8. $(x^{-3}+2)(x^{-3}-2)$.
9. $(x^{-2}-2x^{-1}+1)(x^{-1}-1)$.
10. $(x^{-1}+3)(x^{-2}+x^{-1}-2)$.

Divide:

11. $x^{-2}+2x^{-1}+1$ by $x^{-1}+1$.
12. $x^{-3}-3x^{-2}y^{-1}+3x^{-1}y^{-2}-y^{-3}$ by $x^{-1}-y^{-1}$.
13. $x^{-4}-16y^{-4}$ by $x^{-2}+4y^{-2}$; by $x^{-1}+2y^{-1}$.

Factor:

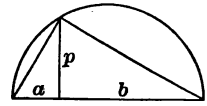
14. $x^{-2}-8x^{-1}+7$.
15. $x^{-2}+6x^{-1}-7$.
16. $2a^{-2}+8a^{-1}+8$.
17. $2a^{-2}+7a^{-1}+6$.

Express in integral form with negative exponents:

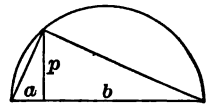
18. $\frac{1}{x}$.
19. $\frac{2}{x^4}$.
20. $\frac{3}{x^2y^2}$.
21. $\frac{5}{x^2y^3}$.
22. $\frac{1}{x^{\frac{1}{2}}}$.
23. $\frac{1}{a^{\frac{3}{4}}}$.
24. $\frac{2}{a^{\frac{3}{4}}}$.
25. $\frac{5}{a^{\frac{1}{3}}}$.
26. $\frac{x^2y^2}{x^3y^3}$.
27. $\frac{x^2y^3}{x^3y^2}$.
28. $\frac{\sqrt{xy}}{xy}$.
29. $\frac{a\sqrt{b}}{b\sqrt{a}}$.
30. $\frac{2}{a^3}-\frac{3}{a^2}+\frac{1}{a}$.
31. $\frac{4}{a^3}+\frac{3}{a^2}-\frac{1}{a}$.
32. $\frac{x^{\frac{1}{2}}}{x}-\frac{y^{\frac{1}{2}}}{y}$.
33. $\frac{x^{\frac{3}{4}}}{x}-\frac{y^{\frac{3}{4}}}{y}$.

165. Graphic Representation of Surds. It is proved in geometry that if, in a semicircle, a perpendicular is drawn from any point on the diameter and extended to meet the circumference, as shown in the figure, then

$$\frac{a}{p} = \frac{p}{b}.$$



Since $\frac{a}{p} = \frac{p}{b}$, it follows that $p^2 = ab$, or $p = \sqrt{ab}$. If we wish to draw a line equal to $\sqrt{3}$, we make $a = 1$, taking any convenient unit, $\frac{1}{4}$ inch being taken here. We then take $b = 3$ and describe the semicircle. Draw the perpendicular p. Then $p = \sqrt{1 \cdot 3} = \sqrt{3}$. By measuring p the length will be found to be about 1.7, the exact length being $\sqrt{3}$, or 1.73 +.



Required to draw a line equal to $\sqrt{5}$.

In the figure, $a = 1$ and $b = 5$. Then $p = \sqrt{1 \cdot 5} = \sqrt{5}$.

Exercise 77. Graphic Representation of Surds

Examples 1 to 7, oral — Examples 8 to 25, written

1. In the above figures, if $a = 1$ and $b = 7$, what does p equal? If $a = 1$ and $b = 9$, what does p equal?

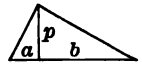
In the above figures find the value of p when:

2. $a = 1, b = 2.$ **5.** $a = 2, b = 3.$ **8.** $a = 7, b = 13.$

3. $a = 1, b = 4.$ **6.** $a = 5, b = 7.$ **9.** $a = 11, b = 21.$

4. $a = 1, b = 6.$ **7.** $a = 7, b = 11.$ **10.** $a = 13, b = 29.$

11. In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, as shown in the figure, then, as in § 165, $\frac{a}{p} = \frac{p}{b}$. If $a = 1.3$ and $b = 4$, what is the length of p?

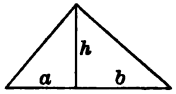


12. In this square, show by § 126 that $d = s\sqrt{2}$.

13. Draw a line $2\sqrt{2}$ inches long, by the method of Ex. 12.

14. In the right triangle below, if $a = 3$ and $b = 4$, what is the value of h? If $a = 4$ and $b = 5$, what is the value of h?

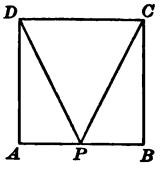
15. In Ex. 14, find the value of h if $a = 5$ and $b = 10$.



16. In Ex. 14, find the value of b if $h = 8$ and $a = 5$.

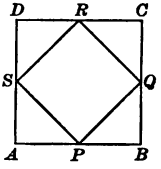
17. In the square below, AP is half of AB. If the side of the square is s, find the length of PC; of $PC + PD$.

In all such cases express the result in the simplest form.



18. In the figure at the right, if $PC = \sqrt{20}$, what is the length of a side of the square?

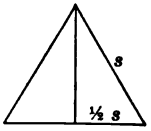
19. In the square below, P, Q, R, and S are midpoints of the sides. If the perimeter of the outer square is 16, what is the perimeter of the inner square?



20. In the figure at the right, if the perimeter of the inner square is 8, what is the perimeter of the outer square?

21. If the side of an equilateral triangle is 2, what is the altitude? If the side is s, what is the altitude?

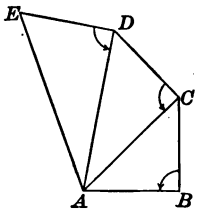
22. If the altitude of an equilateral triangle is h, what is the length of each side? What is the area?



23. If the two sides of a right triangle are $\frac{1}{2}s$ and $\frac{1}{2}s\sqrt{3}$, what is the length of the hypotenuse?

24. If the two sides of a right triangle are x and $\frac{1}{2}x(\sqrt{5}-1)$, what is the length of the hypotenuse?

25. In the figure at the right, if the angles marked by arrows are all right angles, and if $AB = BC = CD = DE = 1$, show that $AC = \sqrt{2}$, $AD = \sqrt{3}$, and $AE = \sqrt{4} = 2$.



166. Equations involving Radicals. Certain forms of equations containing radicals can be solved by the operations already studied. The following are types:

1. Solve the equation $x = 3 + \sqrt{x^2 - 45}$.

Subtracting 3,	$x - 3 = \sqrt{x^2 - 45}$.
Squaring,	$x^2 - 6x + 9 = x^2 - 45$.
Subtracting $x^2 + 9$,	$-6x = -54$.
Dividing by -6,	$x = 9$.

2. Solve the equation $\sqrt{x - 0.1} = \sqrt{x - 3.1} + 1$.

Squaring,	$x - 0.1 = x - 3.1 + 2\sqrt{x - 3.1} + 1$.
Subtracting $x - 3.1 + 1$,	$2 = 2\sqrt{x - 3.1}$.
Dividing by 2,	$1 = \sqrt{x - 3.1}$.
Squaring,	$1 = x - 3.1$.
Adding 3.1,	$4.1 = x$.

Exercise 78. Equations involving Radicals

Solve the following:

1. $7 + 4\sqrt{x} = 79$.
2. $5 + 2\sqrt{x} = 65$.
3. $\sqrt{2x} - 12 = 4$.
4. $\sqrt{x + 2} + 7 = 16$.
5. $\sqrt{x - 5} + 12 = 17$.
6. $\sqrt{x + 12} = 6 - \sqrt{x}$.
7. $\sqrt{x - 7} - \sqrt{x + 1} = -2$.
8. $\sqrt{22 - x} - \sqrt{10 - x} = 2$.
9. $\dfrac{3 - x}{\sqrt{3} - \sqrt{x}} = 7$.
10. $\dfrac{4 - x}{2 - \sqrt{x}} = 9$.
11. $\dfrac{5 - x}{\sqrt{5} - \sqrt{x}} = 8$.
12. $\sqrt{4x} - \sqrt{x} = 7$.
13. $\sqrt{x}(\sqrt{5} + \frac{1}{2}) = 8$.
14. $\sqrt{5x} - \sqrt{\frac{1}{4}x} = 8$.
15. $\sqrt{2x} + \sqrt{x} = 1$.
16. $5\sqrt{x} + \sqrt{3x} = 22$.
17. $2\sqrt{x} - \sqrt{2x} = 2 + \sqrt{2}$.
18. $3\sqrt{x} + \sqrt{5x} = 3 + \sqrt{5}$.
19. $1 + \sqrt{x} = 2(1 - \sqrt{x})$.
20. $\dfrac{3x - 7}{\sqrt{3x} - \sqrt{7}} = \sqrt{7} + 3$.
21. $\dfrac{5x - 9}{\sqrt{5x} - 3} = \sqrt{8} + 3$.
22. $\dfrac{7x - 25}{\sqrt{7x} - 5} = 5 + \sqrt{7}$.

167. Simplification of an Imaginary. Because $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, we see that $\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \cdot \sqrt{-1}$. Therefore an imaginary number (§109) may be written as the product of a real number and $\sqrt{-1}$.

This is advisable in most of the operations with imaginaries.

Imaginaries are met in the solution of certain equations, and therefore a brief discussion is needed at this time. The graphic representation of such numbers is explained in Chapter XIV.

168. Powers of $\sqrt{-1}$. Because $\sqrt{-1}$ enters into all imaginary expressions of the form $\sqrt{-a}$, which equals $\sqrt{a} \cdot \sqrt{-1}$, it is necessary to know the various powers of $\sqrt{-1}$. Since the square of the square root of any number is the number itself, therefore $(\sqrt{-1})^2 = -1$. Then we have

$$(\sqrt{-1})^2 = -1;$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \sqrt{-1} = (-1)\sqrt{-1} = -\sqrt{-1};$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2 (\sqrt{-1})^2 = (-1)(-1) = +1;$$
$$(\sqrt{-1})^5 = (\sqrt{-1})^4 \sqrt{-1} = (+1)\sqrt{-1} = +\sqrt{-1},$$

and so on, repeating the group $\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$.

Frequently i is used for $\sqrt{-1}$, it being the initial of *imaginary*. In this case $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, and so on, the even powers being real and the odd powers imaginary.

169. Product of Two Imaginaries. If the imaginary number $\sqrt{-a}$ is written as above explained, with $\sqrt{-1}$ by itself, the product of such numbers is easily found, thus:

$$\sqrt{-a} \cdot \sqrt{-b} = \sqrt{a} \cdot \sqrt{-1} \cdot \sqrt{b} \cdot \sqrt{-1}$$
$$= \sqrt{a} \cdot \sqrt{b} \cdot (\sqrt{-1})^2 = \sqrt{a} \cdot \sqrt{b} \cdot (-1)$$
$$= -\sqrt{ab}.$$

That is, $$\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{ab}.$$

Similarly, $$(-\sqrt{-2})(-\sqrt{-3}) = (-\sqrt{2})\sqrt{-1}\ (-\sqrt{3})\sqrt{-1}$$
$$= (+\sqrt{6})(\sqrt{-1})^2$$
$$= -\sqrt{6}.$$

170. Complex Number. The sum of a real number and an imaginary number is called a *complex number.*

Thus $2+\sqrt{-3}$, or $2+\sqrt{3}\sqrt{-1}$ is a complex number. In the study of certain equations we need to be able to add and multiply complex numbers.

The general form of a complex number such as we shall meet in our work is $a+b\sqrt{-1}$. Thus $5-\sqrt{-20}=5-\sqrt{4}\sqrt{5}\sqrt{-1}=5-2\sqrt{5}\sqrt{-1}$. Here $a=5$ and $b=-2\sqrt{5}$.

171. Sum of Two Complex Numbers. Complex numbers are added as follows:

$$\begin{array}{r} 3+4\sqrt{-1} \\ 5-7\sqrt{-1} \\ \hline 8-3\sqrt{-1} \end{array} \qquad \begin{array}{r} a+b\sqrt{-1} \\ p+q\sqrt{-1} \\ \hline (a+p)+(b+q)\sqrt{-1} \end{array}$$

Similarly, $3+\sqrt{-8}=3+2\sqrt{2}\sqrt{-1}$

$$-2+3\sqrt{-16}=-2+12\sqrt{-1}$$

$$1+(12+2\sqrt{2})\sqrt{-1}$$

172. Product of Two Complex Numbers. Complex numbers are multiplied (§ 169) as follows:

$$\begin{array}{r} 2+3\sqrt{-1} \\ 5-2\sqrt{-1} \\ \hline 10+15\sqrt{-1} \\ -4\sqrt{-1}+6 \\ \hline 10+11\sqrt{-1}+6 \\ =16+11\sqrt{-1} \end{array} \qquad \begin{array}{r} a+b\sqrt{-1} \\ p+q\sqrt{-1} \\ \hline ap+pb\sqrt{-1} \\ aq\sqrt{-1}-bq \\ \hline ap+(pb+aq)\sqrt{-1}-bq \\ =(ap-bq)+(pb+aq)\sqrt{-1} \end{array}$$

173. Conjugate Complex Numbers. Two numbers of the forms $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are called *conjugate complex numbers.*

Since $(a+b\sqrt{-1})+(a-b\sqrt{-1})=2a,$

and $(a+b\sqrt{-1})(a-b\sqrt{-1})$

$$=a^2-ab\sqrt{-1}+ab\sqrt{-1}+b^2$$

$$=a^2+b^2,$$

the sum or the product of two conjugate complex numbers is a real number.

Exercise 79. Imaginary and Complex Numbers

Examples 1 to 8, oral — Examples 9 to 35, written

1. Express in the form $a\sqrt{-1}$: $\sqrt{-4}$; $\sqrt{-9}$; $\sqrt{-16}$.
2. Express in the form $a\sqrt{-1}$: $\sqrt{-49}$; $\sqrt{-81}$; $\sqrt{-121}$.

Express in the form $a+b\sqrt{-1}$:

3. $2+\sqrt{-25}$.
4. $3-\sqrt{-64}$.
5. $7+\sqrt{-100}$.
6. $3-\sqrt{-400}$.
7. $2+\sqrt{-3}$.
8. $3-\sqrt{-7}$.

Reduce to the form $a\sqrt{-1}$:

9. $2\sqrt{-289}$.
10. $4\sqrt{-361}$.
11. $\frac{1}{2}\sqrt{-441}$.
12. $\frac{1}{4}\sqrt{-784}$.
13. $\frac{1}{9}\sqrt{-81x^2}$.
14. $\sqrt{-a^2b^4c^6}$.
15. $\sqrt{-\frac{1}{9}}$.
16. $\sqrt{-7a^2b^2}$.
17. $\sqrt{-12x^4y^8}$.

Simplify:

18. $\sqrt{-4}+\sqrt{-9}$.
19. $\sqrt{-36}-\sqrt{-25}$.
20. $\sqrt{-144}+\sqrt{-49}$.
21. $\sqrt{-625}-\sqrt{-361}$.
22. $(2+3\sqrt{-1})+(3-7\sqrt{-1})+(-5+2\sqrt{-1})$.
23. $(3-2\sqrt{-8})+(5+7\sqrt{-18})+(7-3\sqrt{-50})$.
24. $(a+b\sqrt{-1})+(b+c\sqrt{-1})+(c+a\sqrt{-1})$.
25. Show that the sum and the product of $5+7\sqrt{-1}$ and $5-\sqrt{-49}$ are both real numbers.

Multiply:

26. $(2+\sqrt{-1})(2-\sqrt{-1})$.
27. $(3+\sqrt{-1})(2-\sqrt{-1})$.
28. $(4+3\sqrt{-1})(2+\sqrt{-1})$.
29. $(3+4\sqrt{-1})(2-\sqrt{-9})$.
30. $(2+3\sqrt{-5})(3-2\sqrt{-5})$.
31. $(4+7\sqrt{-5})(3+6\sqrt{-5})$.
32. $(2\sqrt{3}+5\sqrt{-3})(3\sqrt{3}+2\sqrt{-3})$.
33. $(3\sqrt{5}+2\sqrt{-7})(5\sqrt{5}-4\sqrt{-7})$.
34. Does $x=2\pm\sqrt{-1}$ satisfy the equation $x^2-4x+5=0$?
35. Does $x=-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ satisfy the equation $x^2+x+1=0$?

Exercise 80. Review

Raise to the powers indicated:

1. $(3a^2 - x^3)^5$.
2. $(\frac{1}{2}a^3 - 4x^2)^6$.
3. $(\frac{1}{3}a^2b^2 - 2)^7$.
4. $(\sqrt{x} + 1)^2$.
5. $(\sqrt{xy} + \sqrt{2})^2$.
6. $(x^{\frac{1}{2}}y^{\frac{1}{2}} + z^{\frac{1}{2}})^4$.
7. $(\sqrt{a+b} + 1)^2$.
8. $[(a-b)^{\frac{1}{2}} + 2]^2$.
9. $[x + (2\sqrt{xy} + y)]^2$.

Find the square root of:

10. 12.96.
11. 42.25.
12. 56.25.
13. $36x^2 - 120a^2x - 12a^4x + 100a^4 + 20a^6 + a^8$.

Find the cube root of:

14. 262,144.
15. 884.736.
16. 1481.544.
17. $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.
18. Find the sixth root of $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.
19. Find the eighth root of $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.
20. Reduce $(a^2 + b^2)\sqrt{a+b}$ to an entire surd.
21. Multiply $x + \sqrt{xy} + y$ by $x - \sqrt{xy} + y$.

Rationalize the denominators of the following:

22. $\dfrac{7 + 2\sqrt{x}}{7 - 2\sqrt{x}}$.
23. $\dfrac{5 + \sqrt{2} - \sqrt{3}}{5 + \sqrt{2} + \sqrt{3}}$.
24. $\dfrac{7 + \sqrt{3} - \sqrt{2}}{7 + \sqrt{3} + \sqrt{2}}$.

Find the square root of:

25. $17 - 4\sqrt{13}$.
26. $26 + 6\sqrt{17}$.
27. $49 - 4\sqrt{55}$.

Perform the operations indicated:

28. $(m^{\frac{1}{2}} - m^{\frac{1}{2}}n^{\frac{1}{2}})(m^{\frac{1}{2}} + m^{\frac{1}{2}}n^{\frac{1}{2}})$.
29. $\dfrac{x^2 + xy + y^2}{x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y}$.
30. $(a^{\frac{m}{n}} + b^{-\frac{m}{n}})(a^{\frac{m}{n}} - b^{-\frac{m}{n}})$.
31. $\dfrac{x^{-\frac{5}{2}} + 2x^{-\frac{3}{2}} + 1}{x^{-\frac{1}{2}} + 1}$.

CHAPTER VIII

QUADRATIC EQUATIONS

174. Quadratic Equation. An equation which, when reduced to its simplest form, contains the second power, but no higher power, of an unknown quantity is called a *quadratic equation.*

For example, $x^2 = 9$, $x^2 - 3x = 0$, and $x^2 - 3x + 7 = 0$.

A quadratic equation can always be reduced to the type $ax^2+bx+c=0$, in which a is not zero, but b or c, or both b and c, may be zero.

Thus $3x^2 + 2x = 7$ is a quadratic equation in which $a = 3$, $b = 2$, and $c = -7$; $5x^2 - 2 = 0$ is a quadratic equation in which $a = 5$, $b = 0$, and $c = -2$. In $2x^2 - 3x = 0$, $a = 2$, $b = -3$, and $c = 0$.

175. Coefficients of an Equation. In the equation $ax^2+bx+c=0$, a, b, and c are called the *coefficients of the equation.*

If a, b, and c are numbers expressed by figures, the equation is called a *numerical quadratic;* if some or all are represented by letters, the equation is called a *literal quadratic.* Thus $x^2 + 7x - 3 = 0$ is a numerical quadratic, and $x^2 + ax + m = 5$ is a literal quadratic.

The term represented by c is called the *absolute term*, or the *constant term*, of the equation.

176. Pure Quadratic. If the first power of the unknown quantity is missing, the equation is called a *pure quadratic.*

Thus $x^2 - 4 = 0$ is a pure quadratic.

A pure quadratic is also called an *incomplete quadratic.*

1. Solve the equation $x^2 + 7 = 14$.

Given $x^2 + 7 = 14$.

Subtracting 7, $x^2 = 7$.

Extracting the square root, $x = \pm\sqrt{7}$.

Here the two roots are real but irrational. We may extract the square root of 7 to as many decimal places as we please.

Check. $(\pm\sqrt{7})^2 + 7 = 7 + 7 = 14$.

2. Solve the equation $5x^2 + 15 = 3x^2 + 65$.

Given $5x^2 + 15 = 3x^2 + 65$.

Subtracting 15 and $3x^2$, $2x^2 = 50$.

Dividing by 2, $x^2 = 25$.

Extracting the square root, $x = \pm 5$.

Check. Substitute either $+5$ or -5 for x in the *given* equation.

Then $5 \cdot 25 + 15 = 3 \cdot 25 + 65$,

or $140 = 140$.

When we extracted the square root of x^2 and 25 we might have written the result $\pm x = \pm 5$. In this case we should then have

$$+x = +5, \quad +x = -5, \quad -x = +5, \quad -x = -5.$$

But since we wish only the values of $+x$, we should obtain from $-x = +5$ the value $x = -5$, and from $-x = -5$ the value $x = 5$, which we have already. Therefore *it is unnecessary to write the sign* $\pm$ *before both members.*

Hence there are only two roots. In this case these are real, rational, and numerically equal, but of opposite signs.

Exercise 81. Equations solved by Pure Quadratics

Solve the following equations:

1. $x^2 = 529$.
2. $x^2 = 1369$.
3. $x^2 - 1681 = 0$.
4. $x^2 - 1849 = 0$.
5. $x^2 - 2209 = 0$.
6. $3x^2 - 7803 = 0$.
7. $5x^2 - 16{,}245 = 0$.
8. $2x^2 - 79.38 = 0$.
9. $3x^2 - 151.23 = 0$.
10. $5x^2 - 344.45 = 0$.
11. $x^3 - 14 = 7(7x - 2)$.
12. $(x+1)(x-1) - 120 = 0$.
13. $x^2 + x + 1 = \dfrac{25x - 1}{x - 1}$.
14. $x^2 = 7$.
15. $x^3 = 9x$.
16. $x^2 = 11 + 4\sqrt{7}$.
17. $x^2 = 12 - 6\sqrt{3}$.
18. $x^2 = 36 - 16\sqrt{5}$.
19. $x^2 + 14\sqrt{2} = 51$.
20. $x^2 + 10\sqrt{7} = 32$.
21. $x^2 - 5 = 2\sqrt{6}$.
22. $x^2 - 2\sqrt{77} = 18$.
23. $x^2 - 2\sqrt{91} = 20$.
24. $(x^2 - 4)(x^2 + 4) = -7$.
25. $(x^2 + 1)(x^2 - 1) = 168$.
26. $x + 1 = \dfrac{6 + 4\sqrt{3}}{x - 1}$.

177. Affected Quadratic. A quadratic equation that contains both the second and first powers of the unknown quantity is called an *affected quadratic.*

An affected quadratic is also called a *complete quadratic.*

178. Solving the Quadratic. There are several methods of solving the quadratic equation. One that is quite commonly used is the solution by *completing the square.*

Since $(x \pm a)^2 = x^2 \pm 2ax + a^2$, if we have an equation in the form $x^2 \pm 2ax = b$, we may make the first member a perfect square (*complete the square*) by adding a^2 to both members.

For example, given $x^2 + 6x = 55.$

Adding $(\frac{6}{2})^2$, or 3^2, $x^2 + 6x + 9 = 64.$

Extracting the square root, $x + 3 = \pm 8.$

Therefore $x = -3 \pm 8$

$= 5$ or $-11.$

ax	a^2
x^2	ax
x	a

Check. $5^2 + 6 \cdot 5 = 25 + 30 = 55$, $(-11)^2 + 6 \cdot (-11) = 121 - 66 = 55.$

If the equation has a coefficient for x^2 other than 1, we may reduce the equation to the form $x^2 + 2ax = b$.

For example, given $2x^2 - 3x = 9.$

Dividing by 2, $x^2 - \frac{3}{2}x = \frac{9}{2},$

or $x^2 - 2 \cdot \frac{3}{4}x = \frac{9}{2},$

in which $-\frac{3}{4}$ equals the a of the equation $x^2 + 2ax = b$.

Therefore to solve an affected quadratic:

1. *Reduce the equation to the form* $x^2 + 2ax = b$.
2. *Add to each member the square of half the coefficient of* x.
3. *Then extract the square root of each member.*
4. *Solve the two resulting simple equations.*

In the equation just solved, and in similar cases, we may avoid fractions by multiplying by 4 times the coefficient of x^2, thus:

$$(4x)^2 + 2 \cdot 3 \cdot (4x) = 72,$$

whence $(4x)^2 + 2 \cdot 3 \cdot (4x) + 9 = 81,$

and $4x + 3 = \pm 9,$

which is easily solved. This is sometimes called the *Hindu Method.*

Other methods of solving the quadratic are given on pages 174 and 179.

179. Illustrative Problems. **1.** Solve the equation $3x^2 - 17x + 10 = 0$.

(1) *To solve by factoring, as in § 46.*

Given $3x^2 - 17x + 10 = 0.$

Factoring, $(3x - 2)(x - 5) = 0,$

whence $3x - 2 = 0,$ and $x = \frac{2}{3};$

or $x - 5 = 0,$ and $x = 5.$

(2) *To solve by completing the square, as in § 178.*

Given $3x^2 - 17x + 10 = 0.$

Subtracting 10, $3x^2 - 17x = -10.$

Dividing by 3, $x^2 - \frac{17}{3}x = -\frac{10}{3}.$

Adding $(\frac{17}{6})^2$, $x^2 - \frac{17}{3}x + (\frac{17}{6})^2 = \frac{289}{36} - \frac{10}{3} = \frac{169}{36}.$

Extracting the square root, $x - \frac{17}{6} = \pm\frac{13}{6}.$

Adding $\frac{17}{6}$ and simplifying, $x = 5$ or $\frac{2}{3}.$

(3) *To solve by the Hindu Method.*

Multiplying by $4 \cdot 3$, $(6x)^2 - 34 \cdot (6x) + 120 = 0,$

whence $(6x)^2 - 34 \cdot (6x) = -120.$

Adding 17^2, $(6x)^2 - 34 \cdot (6x) + 17^2 = 169.$

Extracting the square root, $6x - 17 = \pm 13,$

whence $x = 5$ or $\frac{2}{3}.$

2. Solve the equation $x + \frac{m}{x} = n.$

(1) *By clearing of fractions.*

Given $x + \frac{m}{x} = n.$

Multiplying by x and subtracting, $x^2 - nx = -m.$

Adding $\left(\frac{n}{2}\right)^2$, $x^2 - nx + \left(\frac{n}{2}\right)^2 = \frac{n^2 - 4m}{4}.$

Extracting the square root, $x - \frac{n}{2} = \pm\frac{1}{2}\sqrt{n^2 - 4m},$

whence $x = \frac{1}{2}(n \pm \sqrt{n^2 - 4m}).$

(2) *By using negative exponents.*

We have $x + mx^{-1} = n.$

Multiplying by x, $x^2 + m = nx,$

whence the solution can be carried out as in (1).

Exercise 82. Affected Quadratics

Solve the following equations:

1. $x^2 - 28x + 147 = 0$.
2. $x^2 - 14x - 147 = 0$.
3. $x^2 + 28x + 147 = 0$.
4. $2x^2 - 29x + 99 = 0$.
5. $6x^2 - 51x + 99 = 0$.
6. $7x^2 + 4 = 11x$.
7. $14x^2 + 12 = 29x$.
8. $30x^2 + 55 = 91x$.
9. $x^2 + 7x - 9 = 0$.
10. $x^2 + 9x + 15 = 0$.
11. $x^2 + 14 = 3x$.
12. $x(x - 2) = 7$.
13. $2 - x^2 + x = 0$.
14. $3 - x(x - 3) = 4$.
15. $5 + x(x - 7) = 9$.
16. $2x + 4 = 3x^2 + 2$.
17. $x + 36 = x^2 - 36$.
18. $2x^2 + 3x = 7x - 4$.
19. $5x + 3 = x^2 - 9$.
20. $x(x - 2) = 3x + 5$.
21. Find the value of t in the equation $kt + \frac{1}{2}gt^2 = m$.
22. Find the value of K in the equation $K(K + 4) = 77$.

Solve the following equations:

23. $\frac{1}{7}(x - 4) - \frac{2}{5}(x - 2) = (2x + 3) + x$.
24. $\frac{2}{5}(3x^2 - x - 5) - \frac{1}{3}(x^2 - 1) = 2(x - 2)^2$.

25. The sum of the squares of three consecutive numbers is 365. Find the numbers.

26. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.

27. For building 108 rd. of stone wall the contractor would have needed 6 da. less if he had built 3 rd. more per day. How many rods were built a day?

28. The area of a square would be doubled if it were changed into a rectangle 6 in. longer and 4 in. wider. Determine its side.

29. The length of a rectangular field exceeds the breadth by 1 yd., and the area is three acres. Determine the dimensions of the field.

Solve the following equations:

30. $x^2 + ax - 2b = 0$.
31. $x^2 + (a-b)x + ab = 0$.
32. $x^2 = (a+b)x - ab$.
33. $a^2x^2 = b + 2ax$.
34. $m^2x^2 = 2mx - 1$.
35. $y^2 + 4y + 5 = 1 + a^2$.
36. $y^2 + 4ay + 3a^2 = b^2 - a^2$.
37. $w^2 + 6kw - 16k^2 = 0$.
38. $u^2 + 16b^2 = 8bu$.
39. $t^2 + (a-b)t = ab$.
40. $t^2 - ct = c^2d(d-1)$.
41. $a^2v^2 - 2av - 7 = b^2 - 8$.
42. $2(x^2 + ax) = 4 - a^2$.
43. $w^2 - 2a^4 + a^2w = 0$.
44. $x(2a + x) = 24a^2$.
45. $x^2 - 2a(a+b) = (b-a)x$.
46. $t(ab + t) = b(1+a)$.
47. $a(a+8) = 4w(4+w)$.
48. $16x^2 - 9a^2 = 4x + 3a$.
49. $9(w^2 + 3a) = 10a^2w$.
50. $c^2v^2 + c^2v - 3c^2 = 9$.
51. $u^2 + u\sqrt{b} - 6b = 0$.
52. $u^2 + 2u\sqrt{a} = 15a$.
53. $4w^2 + 8w\sqrt{-a} + 5a = 0$.

54. Solve the equation $a^2x^2 + 2ax + 1 = (a+b)^2$, and check the roots. What are the values of the roots if $a = 1$ and $b = 2$?

Solve the following equations:

55. $\dfrac{a}{x} + \dfrac{x}{b} = c$.
56. $\dfrac{a}{x} + \dfrac{x}{a+b} = a$.
57. $\dfrac{a}{2x} + \dfrac{2x}{a} = b$.
58. $ax + \dfrac{a}{x} = a + b$.
59. $ax - \dfrac{a+b}{x} = a$.
60. $\dfrac{a^2}{x^2} + \dfrac{2a}{x} = 3$.
61. $\dfrac{x^2}{a^2} + \dfrac{4x}{a} + 5 = 2$.
62. $\dfrac{(a+b)^2}{x^2} + \dfrac{2(a+b)}{x} = c^2 - 1$.
63. $av - \dfrac{b}{v} = a - b$.
64. $bt - \dfrac{b-1}{t} = b$.
65. $0.36x^2 - 1.2x = a(a+2b) + b^2$.
66. $u(u + 2a - 2b + 2c) = (a + b + 1)(a + b - 1)$.
67. $(a+b)^2v^2 + 2av + 2bv = (c + d - 1)(c + d + 1)$.
68. $(a - b + c)x^2 - 2(a - b + c)dx + d^2 = (a + b - c)^2$.

180. Equations solved like Quadratics. It often happens that an equation is not a quadratic in x, but admits of solution by the aid of quadratics.

1. Solve the equation $\dfrac{(x+2)(x-2)}{5} = \left(\dfrac{3}{x}\right)^2$.

Clearing of fractions, $x^2(x+2)(x-2) = 45$.

Expanding, $x^4 - 4x^2 = 45$.

Completing the square, $x^4 - 4x^2 + 4 = 49$.

Extracting the square root, $x^2 - 2 = \pm 7$.

Solving, $x = \pm 3$ or $\pm\sqrt{-5}$.

There are therefore four roots, two being real and two being imaginary. *An equation of the fourth degree always has four roots.*

2. Solve the equation $x^3 - 1 = 0$.

Factoring, $(x-1)(x^2+x+1) = 0$.

Then the equation is satisfied if either

$$x - 1 = 0,$$

or if $x^2 + x + 1 = 0$ (by § 46).

If $x - 1 = 0$,

$$x = 1.$$

If $x^2 + x + 1 = 0$,

subtracting 1, $x^2 + x = -1$.

Adding $(\frac{1}{2})^2$, $x^2 + x + \frac{1}{4} = -\frac{3}{4}$.

Solving, $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$.

We therefore have three roots, 1, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$. *An equation of the third degree always has three roots.*

3. Solve the equation $77x = x^2(18 - x)$.

Rearranging, $x^3 - 18x^2 + 77x = 0$.

Factoring, $x(x^2 - 18x + 77) = 0$.

Then the equation is satisfied if either $x = 0$, or $x^2 - 18x + 77 = 0$.

Then $x^2 - 18x + 77 = 0$,

and $(x - 11)(x - 7) = 0$.

Solving, $x = 11$ or 7.

Therefore there are three roots, $x = 0$, 11, or 7. *If x is a factor of every term in an equation, one of the roots is zero.*

Exercise 83. Equations solved like Quadratics

Examples 1 to 4, oral — Examples 5 to 23, written

1. If $x^4 = 16$, what are the two values of x^2? From these find the four values of x.

2. If $x^4 + 2x^2 = 3$, what must be added to complete the square? Complete the square and find the values of $x^2 + 1$.

3. What are the two roots of the equation $x^2 - x = 0$?

4. What is one of the roots of the equation $x^3 - x = 0$?

Solve the following equations:

5. $x^4 + 8x^2 = 48$.

6. $y^4 - 6y^2 = 27$.

7. $t^4 - 10t^2 = 31$.

8. $p^4 + 10p^2 = 11$.

9. $q^4 + 7q^2 = 30$.

10. $2m^4 + 3m^2 = 119$.

11. $3(n^4 - 21) = 20n^2$.

12. $4n^2(8n^2 + 3) - 5 = 0$.

13. $x^3 + 1 = 0$.

14. $x^3 - 8 = 0$.

15. $x^3 + 27 = 0$.

16. $x^3 + x = 0$.

17. $8x^3 + 1 = 0$.

18. $K^2(K^2 + 1) = 650$.

19. $T^2(T^2 + 9) = 52$.

20. $2P^4 = 3(45 + P^2)$.

21. If the square of a certain number is multiplied by the square increased by 2, the product is 9. What is the number? Is there any integer that satisfies the conditions? Are there any real numbers? any imaginaries?

22. If the square of a certain number is multiplied by the square increased by 3, the product is 28. What is the number? Check the result. Is the number integral? fractional? real? surd? imaginary?

23. In the equation $x^3 = 1$, x must equal the cube root of 1. This equation, in the form of $x^3 - 1 = 0$, is solved on page 177. It therefore appears that 1 has three cube roots. Show that this statement is true by cubing each of the three roots, 1, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

181. Solution by Formula. Although the methods of factoring and of completing the square are sufficient for solving any quadratic equation, it is convenient to be able to write down the roots at once without the trouble of factoring or completing the square.

Since $3 \times 7 = 7 + 7 + 7$, we could dispense with the multiplication table and perform the multiplications by addition. In the same way we could dispense with the solution of the quadratic by formula and continue to solve by the methods already studied. This is not advisable, however, because it usually takes too long to complete the square or factor.

Every quadratic equation may evidently be reduced to the form $ax^2 + bx + c = 0$, in which a, b, and c are integers.

Given $ax^2 + bx + c = 0.$

Then $ax^2 + bx = -c.$

Dividing by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}.$

Completing the square, $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}.$

Extracting the square root, $x + \frac{b}{2a} = \pm\frac{1}{2a}\sqrt{b^2 - 4ac}.$

Therefore $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

Therefore the roots of an equation in the form $ax^2 + bx + c = 0$ *are*

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $a = 1$, then $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$

Solve the equation $2x^2 - 3x - 9 = 0.$

Here $a = 2$, $b = -3$, $c = -9$.

Substituting these values in the formula,

$$x = \frac{3 \pm \sqrt{9 + 72}}{4}$$

$$= \frac{3 \pm 9}{4} = 3 \text{ or } -\frac{3}{2}.$$

The student may also solve by factoring or by completing the square.

Exercise 84. Solution by Formula

Examples 1 to 4, oral — Examples 5 to 36, written

1. What is the value of $-\frac{b}{2a}$ when $b = -4$, $a = 2$?

2. What is the value of $b^2 - 4ac$ when $b = -4$, $a = 2$, $c = 1$? when $b = 2$, $a = 1$, $c = 1$?

3. When b^2 is greater than $4ac$, are the roots real or are they imaginary?

4. When b^2 is less than $4ac$, are the roots real or are they imaginary?

Solve the following equations, using the formula:

5. $x^2 + 7x + 6 = 0$.

6. $x^2 + 9x + 14 = 0$.

7. $x^2 + x - 12 = 0$.

8. $x^2 - 5x - 6 = 0$.

9. $x^2 + 5x - 6 = 0$.

10. $x^2 - 2x - 15 = 0$.

11. $x^2 - 5x - 14 = 0$.

12. $x^2 - 9x + 14 = 0$.

13. $x^2 + 5x - 14 = 0$.

14. $x^2 - 14x + 33 = 0$.

15. $x^2 - 11x + 28 = 0$.

16. $y^2 - 17y + 72 = 0$.

17. $w^2 - 13w + 42 = 0$.

18. $6x^2 - x - 1 = 0$.

19. $6y^2 - 26y - 1 = 0$.

20. $2t^2 - 5t + 2 = 0$.

21. $5k^2 - 22k + 21 = 0$.

22. $6p^2 + 13p + 6 = 0$.

23. $16m^2 - 49 = 0$.

24. $121x^2 - a^2 = 0$.

25. $a^2x^2 - 4ax - 5 = 0$.

26. $a^2x^2 + 4ax - 21 = 0$.

27. $x^2 - 9x = -9$.

28. $x^2 + 10x = -24$.

29. $x^2 - 2x = 24$.

30. $x^2 + \frac{1}{2}x = \frac{3}{16}$.

31. $a^2x^2 + a(b - c)x - bc = 0$.

32. $x^2 - (a + b)x + ab = 0$.

33. $x^2 + (a - b)x - ab = 0$.

34. $x^2 + (a + 2)x + 2x = 0$.

35. $x^2 + 8x + 49 = 17x + 29$.

36. $3(5x^2 + 12) + 3x = 30 - 13x$.

182. Relation of the Roots to the Coefficients. Designating the two roots of the equation $x^2 + bx + c = 0$ by x_1 and x_2, we have

$$x_1 = \frac{-b + \sqrt{b^2 - 4c}}{2},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Therefore $$x_1 + x_2 = -b,$$

and $$x_1 x_2 = c.$$

For $$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4c}}{2} + \frac{-b - \sqrt{b^2 - 4c}}{2} = \frac{-2b}{2} = -b,$$

and $$x_1 x_2 = \frac{(-b + \sqrt{b^2 - 4c})(-b - \sqrt{b^2 - 4c})}{4}$$

$$= \frac{b^2 - (b^2 - 4c)}{4} = \frac{b^2 - b^2 + 4c}{4} = c.$$

Therefore, in an equation of the form $x^2 + bx + c = 0$,

The product of the roots equals the absolute term.

The sum of the roots equals the coefficient of x with the opposite sign.

Students should be able to apply this check to most equations at sight.

1. Are 3 and -5 the roots of $x^2 + 2x - 15 = 0$?

We see that $3 \cdot (-5) = -15$, the absolute term;

and $3 + (-5) = -2$, minus the coefficient of x.

Therefore 3 and -5 are the roots.

2. Are $-\frac{1}{2}$ and -2 the roots of $2x^2 + 3x - 2 = 0$?

Reduce to the form $x^2 + \frac{3}{2}x - 1 = 0$.

Then $(-\frac{1}{2})(-2) = 1$,

which is not the absolute term (-1). Therefore $-\frac{1}{2}$ and -2 are not the roots.

3. Form the equation whose roots are 6 and $-\frac{2}{3}$.

In the equation $x^2 + bx + c = 0$ we must have

$$b = -(6 - \tfrac{2}{3}) = -5\tfrac{1}{3} = -\tfrac{16}{3},$$

and $$c = 6 \cdot (-\tfrac{2}{3}) = -4.$$

Therefore the equation is $x^2 - \frac{16}{3}x - 4 = 0$, or $3x^2 - 16x - 12 = 0$.

Exercise 85. Relation of Roots and Coefficients

Examples 1 to 17, oral — Examples 18 to 36, written

1. Are 3 and 6 the roots of $x^2 - 9x + 18 = 0$?
2. Are -7 and -2 the roots of $x^2 - 9x + 14 = 0$?
3. What is the sum of the roots of $x^2 + 3 = 2x$?
4. What is the product of the roots of $x^2 + 5x = 6$?

Without solving, determine whether the roots of the following equations are as stated:

5. $x^2 - 8x + 15 = 0$; 5, 3.
6. $x^2 - 16x + 72 = 0$; 7, 9.
7. $x^2 - 14x + 48 = 0$; 6, 8.
8. $x^2 - 15x + 44 = 0$; 4, 11.
9. $x^2 + x - 12 = 0$; 3, -4.
10. $x^2 + 12x + 35 = 0$; 5, 7.
11. $x^2 + 15x + 56 = 0$; 7, 8.
12. $x^2 - 12x + 32 = 0$; 4, 8.
13. $x^2 - 4x + 2 = 0$; 2, 2.
14. $x^2 - 6x - 9 = 0$; 3, -3.
15. $x^2 + x + 1 = 0$; $-\frac{1}{2} + \sqrt{2}$, $-\frac{1}{2} - \sqrt{2}$.
16. $x^2 + (a + b)x + ab = 0$; $-a$, $-b$.
17. $x^2 - (a + b + c)x + (a + b)c = 0$; $-(a + b)$, c.

Form the equations of which the roots are:

18. 2, 3.
19. -2, 3.
20. 2, -3.
21. -2, -3.
22. 7, -13.
23. -11, -17.
24. -19, 35.
25. -55, -13.
26. $\frac{2}{3}$, 9.
27. $\frac{3}{5}$, 15.
28. $\frac{3}{4}$, $\frac{7}{8}$.
29. $-\frac{1}{8}$, 24.
30. a, $-b$.
31. $\frac{1}{2}a$, b.
32. a^3, a^2.
33. $-a^4$, $-b^4$.

34. If some one at the blackboard should give 1.22 and 3.06 as the roots of the equation $x^2 - 4.28x + 3.7147 = 0$, how could you tell, without solving, that they were wrong?

35. If you find $2\frac{1}{2}$ and 3.1416 to be the roots of the equation $x^2 - 5.6416x + 7.854 = 0$, how will you check them?

36. If x_1 and x_2 are the two roots of the equation $x^2 + bx + c = 0$, find the value of $\frac{1}{x_1} + \frac{1}{x_2}$.

183. Nature of the Roots. We have found (§ 181) that the roots of the general quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We see that if $b^2 - 4ac = 0$, the two radicals become zero and the two roots become $-\frac{b}{2a}$ and $-\frac{b}{2a}$; that is, the two roots are real and equal.

For example, in the equation $x^2 - 4x + 4 = 0$ we have $b^2 - 4ac = 16 - 4 \cdot 4 = 0$. Hence we know, before we solve, that the two roots are real and equal.

If $b^2 - 4ac$ is a perfect square, $\sqrt{b^2 - 4ac}$ is rational, and hence both roots are rational.

For example, in the equation $x^2 - 9x + 14 = 0$ we have $b^2 - 4ac = 81 - 56 = 25$. Hence we know, without solving completely, that the two roots are rational, but that they are not equal.

If $b^2 - 4ac$ is positive and not a perfect square, $\sqrt{b^2 - 4ac}$ is irrational, and hence both roots are irrational and unequal.

If $b^2 - 4ac$ is negative, $\sqrt{b^2 - 4ac}$ is imaginary, and hence both roots are imaginary.

184. Discriminant. The expression $b^2 - 4ac$ is called the *discriminant* of the equation $ax^2 + bx + c = 0$.

By its use we discriminate as to the nature of the roots, thus:

If $b^2 - 4ac > 0$, *the roots are real and unequal;*

if $b^2 - 4ac = 0$, *the roots are real and equal;*

if $b^2 - 4ac < 0$, *the roots are imaginary.*

1. What is the nature of the roots of $x^2 - 2x + 1 = 0$?

We have $b^2 - 4ac = (-2)^2 - 4 \cdot 1 \cdot 1 = 4 - 4 = 0$. Hence the discriminant equals 0, and the roots are real and equal. Indeed, we know by inspection that the roots are 1 and 1.

2. What is the nature of the roots of $x^2 - x + 1 = 0$?

We have $b^2 - 4ac = (-1)^2 - 4 \cdot 1 \cdot 1 = 1 - 4 = -3$. Hence the discriminant is negative, and the roots are both imaginary. If we solve, we shall find that the roots are $\frac{1}{2}(1 \pm \sqrt{-3})$.

Exercise 86. Nature of the Roots

Examples 1 to 6, oral — Examples 7 to 26, written

1. What is the discriminant of $x^2 + 2x + 1 = 0$?

2. What is the discriminant of $x^2 + 2x + 2 = 0$?

3. If the discriminant of an equation equals $\frac{2}{3}$, what is the nature of the roots?

4. If the discriminant of an equation equals -0.5, what is the nature of the roots?

5. If the discriminant of an equation equals 0.25, what is the nature of the roots?

6. In the equation $ax^2 + bx + c = 0$, what is the nature of the roots when $b^2 = 4ac$?

Without solving, determine the nature of the roots:

7. $x^2 + x + 1 = 0$.
8. $x^2 + 2x + 3 = 0$.
9. $x^2 - x + 10 = 0$.
10. $x^2 - 3x + 1 = 0$.
11. $x^2 + 4x + 4 = 0$.
12. $x^2 - 2x + 4 = 0$.
13. $x^2 - 2x - 4 = 0$.
14. $x^2 + 7x + 12 = 0$.
15. $5x^2 - 3x - 2 = 0$.
16. $4x^2 - 4x - 1 = 0$.
17. $7x^2 + 9x - 10 = 0$.
18. $3x^2 - 7x - 6 = 0$.
19. $2x^2 - 13x + 15 = 0$.
20. $9x^2 + 12x + 4 = 0$.

21. Give to a, b, and c values that will make $b^2 - 4ac = 0$. Write an equation having two real and equal roots.

22. Give to a, b, and c values that will make $b^2 - 4ac > 0$. Write an equation having two real and unequal roots.

23. Write an equation having two unequal surd roots.

24. Write an equation having two imaginary roots.

25. Write an equation that will have $-2 + \sqrt{-2}$ and $-2 - \sqrt{-2}$ for its roots.

26. For what value of a will the roots of $2x^2 + (1+a)x + 2 = 0$ be equal? real? imaginary?

185. Special Forms. A few other special forms of equations will now be considered.

1. Solve the equation $x^{2n} + 2x^n - 15 = 0$.

This is a quadratic in x^n. Solving for x^n, we have

$$x^n = \frac{-2 \pm \sqrt{4+60}}{2} = \frac{-2 \pm 8}{2} = 3 \text{ or } -5.$$

$$\therefore x = \sqrt[n]{3} \text{ or } \sqrt[n]{-5}.$$

2. Solve the equation $2x^{\frac{2m}{n}} - 17x^{\frac{m}{n}} + 35 = 0$.

Solving for $x^{\frac{m}{n}}$, $x^{\frac{m}{n}} = \frac{17 \pm \sqrt{17^2 - 4 \cdot 2 \cdot 35}}{2 \cdot 2} = \frac{17 \pm \sqrt{289 - 280}}{4}$

$$= \frac{17 \pm 3}{4} = 5 \text{ or } 3\tfrac{1}{2}.$$

Raising each member to the nth power and then extracting the mth root,

$$x = \sqrt[n]{5^m} \text{ or } \sqrt[n]{(3\tfrac{1}{2})^m}.$$

3. Solve the equation $(x^2 + 2x)^2 - 2(x^2 + 2x) - 3 = 0$.

Solving for $x^2 + 2x$, $x^2 + 2x = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1.$

We now have $x^2 + 2x = 3$,

and $x^2 + 2x = -1$.

Solving the first, we have $x = -3 \text{ or } 1$.

Solving the second, we have $x = -1 \text{ or } -1$.

Therefore the roots are $-3, 1, -1, -1$.

4. Solve the equation $x - 8\sqrt{x+3} + 18 = 0$.

We may write this $x + 3 - 8\sqrt{x+3} + 15 = 0$.

Here $x + 3$ is the square of $\sqrt{x+3}$. Therefore, solving for $\sqrt{x+3}$, we have

$$\sqrt{x+3} = 3 \text{ or } 5.$$

Therefore $x + 3 = 9 \text{ or } 25$,

and $x = 6 \text{ or } 22$.

5. Solve the equation $x^{-2} + 3x^{-1} - 4 = 0$.

This is a quadratic in x^{-1}. Solving, $x^{-1} = -4$ or 1. Since $\frac{1}{x} = -4$ or 1, therefore $x = -\frac{1}{4}$ or 1.

Exercise 87. Special Forms of Quadratics

Examples 1 to 3, oral — Examples 4 to 25, written

1. In the equation $x^{2n} + bx^n + c = 0$, what is the sum of the two values of x^n? What is their product?

2. In the equation $x - 2x^{\frac{1}{2}} + 1 = 0$, how is x related to $x^{\frac{1}{2}}$? For what unknown quantity shall you solve the quadratic?

3. In the equation $x^{\frac{m}{n}} - 2x^{\frac{m}{2n}} + 1 = 0$, how is $x^{\frac{m}{n}}$ related to $x^{\frac{m}{2n}}$? For what unknown quantity shall you solve the quadratic?

Solve the following equations:

4. $x - 8\sqrt{x} + 15 = 0$.
5. $x - 10\sqrt{x} + 21 = 0$.
6. $x - 13x^{\frac{1}{2}} + 40 = 0$.
7. $x^3 - 6x^{\frac{3}{2}} + 8 = 0$.
8. $x^8 - 20x^4 + 99 = 0$.
9. $x^p - 15x^{\frac{p}{2}} + 56 = 0$.
10. $x^{2p} - 13x^p + 30 = 0$.
11. $x^{-2} - 12x^{-1} + 27 = 0$.
12. $x^{-2} - 9x^{-1} + 8 = 0$.
13. $9x^{-2} - 21x^{-1} + 12 = 0$.
14. $(x^2 - 3x)^2 + 3(x^2 - 3x) + 2 = 0$.
15. $(x^2 + 7x)^2 + 5(x^2 + 7x) - 84 = 0$.
16. $(x^2 + 5x + 1)^2 + 4(x^2 + 5x + 1) + 5 = 0$.
17. $x - 9 + 9\sqrt{x - 9} + 20 = 0$.
18. $x^2 + 3x + 2 + 11\sqrt{x^2 + 3x + 2} + 30 = 0$.
19. $(3x - 5)^2 - 8(3x - 5) + 7 = 0$.
20. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}$.
21. $3\sqrt{3x^2 - 2x + 4} = 3x^2 - 2x - 6$.
22. $39 - 8x + \sqrt{7x^2 + 8x - 19} - 7x^2 = 0$.
23. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3$.
24. $2x^2 - 6x + \sqrt{x^2 - 3x + 6} - 9 = 0$.
25. In the equation $\frac{x^2}{x+1} + \frac{x+1}{x^2} = \frac{5}{2}$, let $y = \frac{x^2}{x+1}$ and solve for y. Then let $\frac{x^2}{x+1}$ equal the results and solve for x.

186. Extraneous Roots. Sometimes in working with equations a root appears that cannot be checked.

For example, if we have the equation $x - 2 = 5$, and we square both members, we have $x^2 - 4x + 4 = 25$,

or $x^2 - 4x - 21 = 0$.

Solving, $x = 7$ or -3.

Of these roots, 7 can be checked because we started with it, and $7 - 2 = 5$; but -3 cannot be checked, for $-3 - 2$ does not equal 5.

Therefore -3 is called an *extraneous root*. This is only another way of saying that -3 is not a root of the original equation, although it is a root of some of the derived equations.

If both members of an equation are raised to the same power, an extraneous root is likely to appear.

It is therefore necessary to check with great care the roots of any equation involving radicals.

Solve $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$.

Squaring, $x + 3 + 2\sqrt{(x+3)(x+8)} + x + 8 = 25x$.

Hence, $2\sqrt{x^2 + 11x + 24} = 23x - 11$.

Squaring, $4x^2 + 44x + 96 = 529x^2 - 506x + 121$.

Hence, $525x^2 - 550x + 25 = 0$.

Dividing by 25, $21x^2 - 22x + 1 = 0$.

Solving, $x = 1$ or $\frac{1}{21}$.

Of these roots 1 will check and $\frac{1}{21}$ will not. Hence $\frac{1}{21}$ is an extraneous root.

It must be remembered that the radical sign indicates only the principal root (§ 116), and hence we cannot say that

$$\sqrt{64} + \sqrt{169} = -8 + 13 = 5.$$

If both members of an equation are multiplied by some function of the unknown quantity, an extraneous root is likely to appear.

For example, if we multiply $x - 5 = 0$ by $x - 2$, we have $(x-5)(x-2) = 0$. The roots of the latter are 5 and 2. Hence an extraneous root, 2, has been introduced.

In general, in clearing of fractions extraneous roots are not introduced, but it is necessary to check to make sure of this fact.

Exercise 88. Radical Equations

Solve the following equations:

1. $x^2+\sqrt{x^2-7}=19.$
2. $x^2+x^{-2}+x+x^{-1}=4.$
3. $\dfrac{1}{(3+x)^{-\frac{1}{2}}}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}.$
4. $x^2+x+\frac{1}{6}(x^2+x)^{\frac{1}{2}}=\frac{7}{6}.$
5. $(x+1)^{\frac{1}{2}}+(x-1)^{\frac{1}{2}}=5.$
6. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$
7. $15x-3x^2+4\sqrt{x^2-5x+5}=16.$
8. $x+2-4x\sqrt{x+2}=12x^2.$
9. $\sqrt{9x^2+21x+1}-\sqrt{9x^2+6x+1}=3x.$
10. $x^{\frac{1}{3}}-4x^{\frac{2}{3}}+x^{-\frac{1}{3}}+4x^{-\frac{2}{3}}+1=-\frac{3}{4}.$
11. $\sqrt{2x+1}-\sqrt{x+4}=\frac{1}{3}\sqrt{x-3}.$
12. $\sqrt{(x-a)^2+2ab+b^2}+2a+x=2x+a+b.$
13. $\sqrt{(x+a)^2+2ab+b^2}+2b+3a+4x=3b+2a+3x.$
14. $4x^{\frac{1}{2}}-3(x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-2)+x^{\frac{1}{2}}=x^{\frac{1}{2}}(11-3x^{\frac{1}{2}}).$
15. $(x^{\frac{1}{2}}-2)(x^{\frac{1}{2}}-4)+72=x^{\frac{1}{2}}(x^{\frac{1}{2}}-1)^2+60.$
16. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}.$
17. $\dfrac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7.$
18. $x+\sqrt{x}=12.$
19. $x-2\sqrt{x}=8.$
20. $2x+3\sqrt{x}=65.$
21. $3x-7\sqrt{x}=66.$

22. A certain number increased by twice its square root equals 15. What is the number?

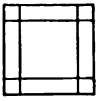
23. Three times a certain number is decreased by five times its square root, the result being 112. Find the number.

24. The square of a certain number is decreased by 55, the result being minus the square root of the square of the number decreased by 13. Find the number.

25. If to the square of a certain number there is added the square root of 17 less than the square of the number, the result is 89. Find the number.

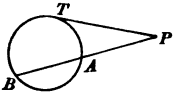
Exercise 89. Problems involving Quadratics

1. A square piece of tin is made into a box by cutting from the corners small squares 2 in. on a side. The box then contains 50 cu. in. Required the dimensions of the piece of tin.

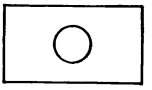


2. A square piece of tin is made into a box by cutting from the corners small squares n inches on a side. The box then contains v cubic inches. Required the dimensions of the piece of tin.

3. It is proved in geometry that $\overline{PT}^2 = PA \cdot PB$, PT being a tangent and PB any line from P meeting the circle twice. If it is known that $PA = 2$, and $PB = 1 + PT$, what is the length of PT? If it is known that $PA = 3$, and $PB = 2 + PT$, what is the length of PT?



4. In a steel plate to be used in a steamship is a porthole 14 in. in diameter. The plate is 5 ft. longer than wide. The area of the plate, exclusive of the porthole, is $34\frac{9}{7}\frac{7}{2}$ sq. ft. What are the dimensions of the plate? (Take $3\frac{1}{7}$ for π.) If the diameter of the porthole is 12 in. and the area of the plate, exclusive of the porthole, is $23\frac{3}{1}\frac{}{4}$ sq. ft., what are the dimensions?



5. The circle shown below is so drawn as just to touch the four sides of the square. The area of that part of the square not covered by the circle is $10\frac{1}{2}$ sq. in. What is the radius of the circle?

6. In a certain trapezoid the upper base is 2 in. and the lower base equals the altitude. The area of the trapezoid is $31\frac{1}{2}$ sq. in. Find the length of the lower base.

7. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length equals the sum of the width and thickness. The number of cubic yards in its volume, added to the number of linear yards in its edges, is $\frac{5}{3}$ of the number of square yards in its surface. Required the dimensions.

8. It is shown in physics that if a body is thrown downward with an initial velocity of V feet per second, the distance d feet that it will pass through in t seconds is given by the formula $d = Vt + 16t^2$. If a body is thrown downward from a height of 750 ft. with an initial velocity of 48 ft. per second, how long will it take it to reach the ground?

9. In Ex. 8, if there is no initial velocity, that is, if the body is simply dropped, the formula becomes $d = 16t^2$. A stone is dropped into a vertical shaft of a mine, and the sound of its striking is heard 10 seconds later. If the velocity of sound is 1120 ft. per second, how deep is the shaft?

10. In Ex. 8, if the body is thrown vertically upward instead of downward, the formula is $d = Vt - 16t^2$. A boy shoots an arrow vertically upward with an initial velocity of 96 ft. per second. How long before it is 80 ft. above the ground? Show that both results are admissible.

11. The perimeter of a right triangle is 108 in. and the hypotenuse is 45 in. Find the length of each side.

12. The area of a right triangle is 384 sq. in. and the hypotenuse is 40 in. Find the length of each side.

13. The hypotenuse of a right triangle is 7 in. longer than one side and 14 in. longer than the other side. Find the length of the hypotenuse and of the two sides.

14. A certain number of students pay \$20 a week for table board at a boarding house. They find that by preparing their own meals they can buy the food for themselves and one other student for \$10 a week, each of the group saving \$3 a week on what would be paid at the boarding house. How much does each pay if they prepare their own meals?

15. A boy was sent to market with \$1.50 to buy some oranges. The price had been raised 10¢ a dozen, so that he bought half a dozen less than he expected to buy. Find the former price, the present price, and the number of oranges purchased.

16. If the edges of a rectangular box are increased by 2 in., 3 in., and 4 in. respectively, the box becomes a cube and its capacity is increased by 1008 cu. in. Find the dimensions of the box.

17. A railway train makes a run of 799 mi. in a certain time. If the rate of speed should be reduced $4\frac{1}{2}$ mi. an hour, the run would take 108 min. longer. How long does it take the train to make the run?

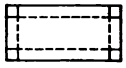
18. A reservoir can be filled by two pipes, A and B, in 9 min., when both are open. The pipe A can fill it in 24 min. less time than B. How long will it take A to fill it?

19. A and B start on two roads that cross at right angles, at distances of 20 and 24 miles respectively from the crossing, and walk toward it at the same rate. How far must they go before the distance between them is reduced 4 mi.?

20. A certain rectangle is 1.3 ft. longer than wide. If the length is increased by 0.6 ft. and the width decreased by 0.1 ft., the area is increased by 0.86 sq. ft. Required the original dimensions.

21. A box containing 384 cu. in. is made by cutting out the corners of a square piece of pasteboard and then turning up each side 6 in. What was the area of the original square?

22. A box containing 324 cu. in. is made by cutting out the corners of a rectangular piece of pasteboard that was half as wide as long, and then turning up each side 6 in. What were the dimensions of the rectangle?



23. A string lies on the circumference of a circle and exactly equals it in length. It is then stretched out to form a rectangle of which the length is 2 in. more than the width, and of which the area is 120 sq. in. Find the dimensions of the rectangle, the circumference of the circle, the radius of the circle, and the area of the circle. (Take $3\frac{1}{7}$ for π.)

24. The denominator of a certain fraction exceeds twice the numerator by 3. If $3\frac{3}{4}$ is added to the fraction, the result is the reciprocal of the original fraction. Find the original fraction.

25. A broker sells a number of railway shares for \$3240. A few days later, the price having fallen \$9 a share, he buys, for the same sum, five more shares than he sold. Find the number of shares transferred on each day, and the price paid.

26. The length of a rectangular lot exceeds its breadth by 20 yd., and the area is 4800 sq. yd. Find the dimensions.

27. Twice the breadth of a rectangular lot exceeds the length by 2 yd., and the area of the lot is 1200 sq. yd. Find the dimensions.

28. A farmer sold a horse for \$138 and gained on the cost one eighth as many per cent as the number of dollars that the horse cost. Find the cost of the horse.

29. A certain train leaves A for B, the distance being 216 mi. Three hours later another train leaves A on the same line and travels 8 mi. an hour faster than the first, arriving at B 45 min. behind it. Find the time taken by each train.

30. A cistern with a capacity of 900 gal. can be filled by two pipes running together in as many hours as the larger pipe brings in gallons per minute. The smaller pipe brings in 1 gal. less per minute than the large one. How long will it take each pipe by itself to fill the cistern?

31. A number is formed by two digits, the units' being less by 3 than half the square of the tens'. If 9 is added to the number, the order of the digits is reversed. Find the number.

32. If the square root of forty-two times a certain positive integer is divided by the integer, the result is the square root of the next larger integer. Find the number.

33. Is it possible to find four consecutive positive integers such that the quotient of any two equals the quotient of the other two? Prove it.

CHAPTER IX

SIMULTANEOUS QUADRATIC EQUATIONS

187. Simultaneous General Quadratics. The most general type of quadratic equation with two unknown quantities contains both of these quantities in the first and second degrees, contains their product, has an absolute term, and has coefficients for all the terms containing an unknown quantity.

That is, just as $ax^2 + bx + c = 0$ is the most general quadratic equation with one unknown quantity, so $ax^2 + bxy + cy^2 + dx + ey + f = 0$ is the most general quadratic equation with two unknown quantities.

Two such equations may be represented by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

and $$a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0.$$

If we eliminated y, we should have an equation of the fourth degree in x that could not be solved by quadratics.

Therefore, *in general, two simultaneous quadratic equations cannot be solved by the method of quadratics.*

There are, however, certain special types that can be solved, and these are considered in this chapter.

That the student may see the difficulties, even in the case of two quadratic equations that seem easy of treatment, consider the system

$$x^2 + y = 7, \tag{1}$$

$$x + y^2 = 11. \tag{2}$$

From (1), $$y = 7 - x^2. \tag{3}$$

Substituting (3) in (2), $$x + (7 - x^2)^2 = 11,$$

whence $$x^4 - 14x^2 + x + 38 = 0.$$

This is an equation of the fourth degree, and although one root ($x = 2$) may easily be found by factoring, it cannot be obtained from this equation by the methods of quadratics.

188. Simple and Quadratic Equations. Before considering two simultaneous quadratic equations we shall consider two simultaneous equations of which one is simple and the other quadratic.

Solve the equations

$$2x^2 + y^2 = 33 \quad (1)$$

$$2x + y = 9 \quad (2)$$

Subtracting $2x$ from (2), $y = 9 - 2x.$ (3)

Substituting (3) in (1), $2x^2 + (9 - 2x)^2 = 33,$

whence $x^2 - 6x + 8 = 0.$

Solving, $x = 2$ or $4.$

Substituting in (3), $y = 5$ or $1.$

We therefore have two pairs of roots, in which

$x = 2$ when $y = 5,$

and $x = 4$ when $y = 1.$

For the first pair

equation (1) becomes $2 \cdot 2^2 + 5^2 = 8 + 25 = 33,$

equation (2) becomes $2 \cdot 2 + 5 = 4 + 5 = 9,$

and similarly for the second pair.

In substituting, join with x the corresponding value of y.

That is, $x = 2$ will not check with $y = 1$, nor $x = 4$ with $y = 5$.

Exercise 90. Simple and Quadratic Equations

Solve the following equations:

1. $x - y = 14$
   $x^2 + y^2 = 436$
2. $x + y = 12$
   $x^2 + y^2 = 104$
3. $x + y = 49$
   $x^2 + y^2 = 1681$
4. $x - 3y = 1$
   $xy + y^2 = 5$
5. $4y = 5x + 1$
   $2xy = 33 - x^2$
6. $x - y = 1$
   $x^2 + y^2 = 2\frac{1}{6}\,xy$
7. $x - y = 2$
   $15(x^2 - y^2) = 16xy$
8. $x + y = a$
   $x^2 + y^2 = b^2$
9. $5x + 2y = 7$
   $7x^2 - 8xy = 159$
10. $x + y = 2$
    $x^2 - 2xy - y^2 = 1$

189. Graph of a Quadratic Equation. To understand more clearly the relation of a linear to a quadratic equation it is helpful to study the graph of each. Since we have already studied the graphs of linear equations, it is only necessary at this time to consider the graph of the quadratic.

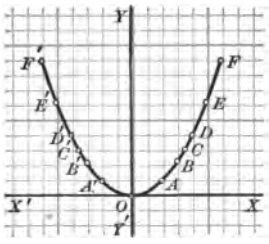
1. Plot the equation $x^2 = 4y$.

If y is negative, x is imaginary and there are no points. We therefore take positive values of y.

If $y =$	0	1	2	3	4	$6\frac{1}{4}$	9
Then $x =$	0	± 2	± 2.8	± 3.5	± 4	± 5	± 6
Points =	O	A, A'	B, B'	C, C'	D, D'	E, E'	F, F'

Plot the several points A, A', B, B', $\cdots$. Through these draw the smooth curve from O to F and F'. This curve is the graph of the equation $x^2 = 4y$. Of course we might take larger values for y and find the corresponding values for x, it being evident that the curve is infinite in extent.

This curve is called a *parabola*. The graph of a quadratic equation containing the first power of one variable and the second power with or without the first power of the other variable is always a parabola. In particular, $x^2 = ay$ and $y^2 = bx$ are both parabolas. If we should plot the latter equation ($y^2 = bx$), we would find that it lies to the right of YY', in the first and fourth quadrants.



In the equation $x^2 = 4y$, when $y = 0$, $x^2 = 0$. The two values of x are then both equal to 0. We measure the values of x on the axis of x. If, therefore, we have the equation $x^2 = 0$, the two roots are equal, and each is 0.

In the same equation ($x^2 = 4y$), when $y = 1$, $x^2 = 4$. The two values of x are then ± 2, and are the abscissas of the points $(2, 1)$, $(-2, 1)$. Similarly, we can solve and find the two values of x corresponding to any value of y by simply measuring the abscissas. By the help of coördinate paper this is easily done.

2. Plot the equation $x^2 + y^2 = 25$.

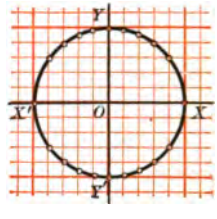
We may write this $y^2 = 25 - x^2$.

If $x^2 > 25$, or if $x > 5$, y is imaginary, and similarly if $x < -5$. We will first take for x values from -4 to $+4$.

If $x =$	-4	-3	-2	-1	0	1	2	3	4
Then $y =$	± 3	± 4	± 4.6	± 4.9	± 5	± 4.9	± 4.6	± 4	± 3

Furthermore, if $x = \pm 5$, $y = 0$, which gives two other points.

Plotting the points and drawing the smooth curve as before, the graph is seen to be a *circle* (circumference). The graph of a quadratic equation in the form $x^2 + y^2 = r^2$ is always a circle.



There is a more general form for the equation of a circle, $x^2 + y^2 + ax + by + c = 0$, where a, b, and c are any given numbers.

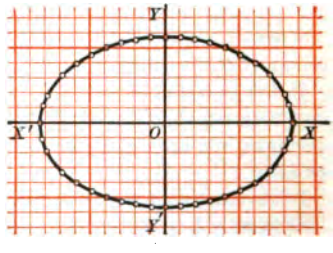
3. Plot the equation $4x^2 + 9y^2 = 288$.

We may write this $y^2 = 32 - \frac{4}{9}x^2$.

If $\frac{4}{9}x^2 > 32$, or if $\frac{2}{3}x > \sqrt{32}$, or $\frac{2}{3}x < -\sqrt{32}$, y is imaginary; that is, x cannot be greater than $\frac{3}{2}\sqrt{32}$, or about 8.5, and x cannot be less than -8.5.

If $x =$	± 8.3	± 8	± 7	± 6	± 5	± 4	± 3	± 2	± 1
Then $y =$	± 1.2	± 1.9	± 3.2	± 4	± 4.6	± 5.0	± 5.3	± 5.5	± 5.6

Furthermore, if $x = \pm 8.5$, $y = 0$; and if $x = 0$, $y = \pm 5.7$.



It will be seen that $+8.3$ corresponds both to $+1.2$ and to -1.2, and that -8.3 also corresponds both to $+1.2$ and to -1.2.

Plotting the points and drawing the curve, the graph is an *ellipse*. The graph of a quadratic equation in the form $ax^2 + by^2 = c$ is always an ellipse, a, b, and c having any values, and a and b having like signs. For example, $2x^2 + 3y^2 - 18 = 0$ is the equation of an ellipse.

4. Plot the equation $xy = 5$.

We may write this $y = \frac{5}{x}$. We then prepare these two tables:

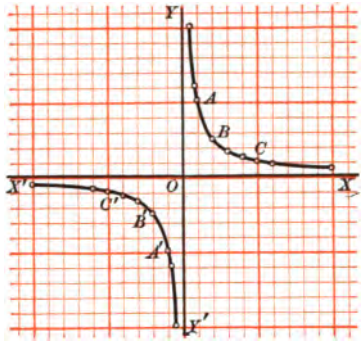
If $x =$	$\frac{1}{2}$	$\frac{5}{6}$	1	2	3	4	5	6	10
Then $y =$	10	6	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1	$\frac{5}{6}$	$\frac{1}{2}$

If $x =$	$-\frac{1}{2}$	$-\frac{5}{6}$	-1	-2	-3	-4	-5	-6	-10
Then $y =$	-10	-6	-5	$-2\frac{1}{2}$	$-1\frac{2}{3}$	$-1\frac{1}{4}$	-1	$-\frac{5}{6}$	$-\frac{1}{2}$

The graph is in this case an *hyperbola*. An equation in the form $xy = a$ is always the equation of an hyperbola.

The three curves already mentioned — the parabola, ellipse, and hyperbola — are called *conic sections* or *conics*. The circle may be considered a special form of the ellipse.

The graph of a quadratic equation is always a conic or a pair of straight lines.

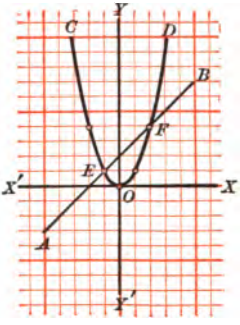


190. Graph of a Linear and Quadratic Equation. We see therefore that the roots of a linear and quadratic equation may be found by the intersection of graphs.

For example, solve the equations

$$x^2 = y$$
$$x - y + 2 = 0$$

Here the graph of $x^2 = y$ is the parabola CD, and the graph of $x - y + 2 = 0$ is a straight line AB. Their common points, E and F, are $(-1, 1)$ and $(2, 4)$. Hence the common values of x and y are $x = -1, +2$, $y = +1, +4$.



Exercise 91. Graphs

Examples 1 to 5, oral — Examples 6 to 29, written

1. What kind of graph has the equation $x + 4y = 71$?
2. What kind of curve is represented by $x^2 + y^2 = 25$?
3. What kind of conic is represented by $y^2 = 2x$?
4. What kind of conic is represented by $x^2 + 3y^2 = 20$?
5. What kind of conic is represented by $xy = 16$?

Plot the following equations:

6. $x - 3y = 5$.
7. $x^2 + y^2 = 4$.
8. $x^2 + 9y^2 = 25$.
9. $x^2 = 8y$.
10. $xy = 9$.
11. $x^2 + 2x = y$.
12. $x^2 - 5x = y$.
13. $y = x^2 + 3x + 4$.

Find by graphs the roots of the following by letting the first member equal y, and then considering the graph when $y = 0$:

14. $x^2 - 3x + 2 = 0$.
15. $x^2 - 5x + 6 = 0$.
16. $x^2 + x - 12 = 0$.
17. $x^2 + x - 30 = 0$.
18. $x^2 - 15x + 56 = 0$.
19. $x^2 + 4x - 77 = 0$.

Solve exactly or approximately by the use of graphs:

20. $x^2 + y^2 = 61$
    $3x - 2y = 3$
21. $x^2 + y^2 = 36$
    $5x + y = 23$
22. $x^2 + y^2 = 100$
    $3x + 4y = 50$
23. $x^2 + 9y^2 = 81$
    $x - 3y = 5$
24. $xy = 30$
    $x + y = 11$
25. $xy = 12$
    $x - y = 1$
26. $x^2 = 9y$
    $x + 2y = 6$
27. $xy = 25$
    $x + 2y = 15$
28. $x^2 = 9y$
    $2x + y = 16$
29. $x^2 + x + 7 = y$
    $4x - y + 4 = -1$

191. Two Quadratics, One being Homogeneous. Given two simultaneous quadratics, in one of which the terms are all of the second degree, a solution is always possible.

Solve the equations

$$x^2 + xy - 6y^2 = 0 \quad (1)$$

$$x^2 + 4x + 3y = 69 \quad (2)$$

From (1), $x = \dfrac{-y \pm \sqrt{y^2 + 24y^2}}{2} = \dfrac{-y \pm 5y}{2} = 2y$ or $-3y$.

Substituting $2y$ for x in (2),

$4y^2 + 8y + 3y = 69$,

or $4y^2 + 11y = 69$.

Solving, $y = 3$ or $-\frac{23}{4}$.

$\therefore x = 2y = 6$ or $-\frac{23}{2}$.

Substituting $-3y$ for x in (2),

$9y^2 - 12y + 3y = 69$,

or $y^2 - y = \frac{23}{3}$.

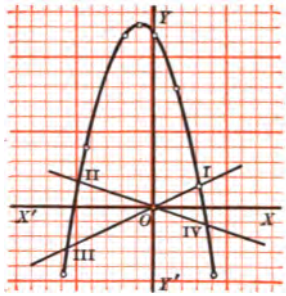
Solving, $y = \dfrac{3 \pm \sqrt{285}}{6}$.

$\therefore x = -3y = \dfrac{-3 \mp \sqrt{285}}{2}$.

Therefore, when $x = 6, \ \dfrac{-3-\sqrt{285}}{2}, \ -\dfrac{23}{2}, \ \dfrac{-3+\sqrt{285}}{2}$,

the corresponding value of $y = 3, \ \dfrac{3+\sqrt{285}}{6}, \ -\dfrac{23}{4}, \ \dfrac{3-\sqrt{285}}{6}$.

If we draw the graph of these two equations, we shall find that the first equation represents two intersecting straight lines, and the second one a parabola.



For (1) reduces to two linear equations, $x = 2y$ and $x = -3y$. These have for their graphs the two straight lines.

In (2),

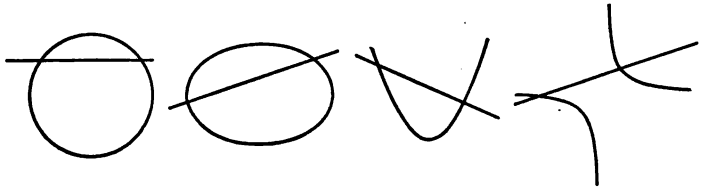
If $x =$	-12	-9	-4	-2	0	3	6	8	9
Then $y =$	-9	8	23	$24\frac{1}{3}$	23	16	3	-9	-16

Plotting these points and drawing the curve, letting each square represent two units, we have the parabola. The four pairs of roots given above are represented on the figure by I, II, III, IV, in order.

Exercise 92. Simultaneous Quadratics

Examples 1 to 10, oral — Examples 11 to 18, written

1. In how many points can a straight line intersect a circle? intersect an ellipse? a parabola? a hyperbola?



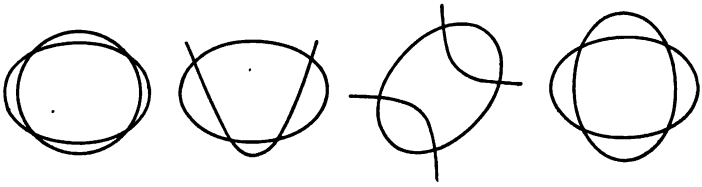
2. How many roots in the system $x + 5y = 9$, $x^2 + y^2 = 25$?

3. How many roots in the system $x - 4y = 1$, $x^2 + 3y^2 = 8$?

4. How many roots in the system $x = 3y$, $y^2 = 8x$?

5. How many roots in the system $3x + y = 15$, $xy = 1$?

6. In how many points can an ellipse intersect a circle? intersect a parabola? a hyperbola? another ellipse?



7. How many roots in the system $x^2 + 7y^2 = 12$, $x^2 + y^2 = 8$?

8. How many roots in the system $x^2 + 6y^2 = 25$, $x^2 = y + 5$?

9. How many roots in the system $x^2 + 6y^2 = 25$, $xy = 3$?

10. How many roots in the system $5x^2 + y^2 = 25$, $x^2 + 5y^2 = 25$?

Plot the equations in the following examples:

11. Ex. 2. **13.** Ex. 4. **15.** Ex. 7. **17.** Ex. 9.

12. Ex. 3. **14.** Ex. 5. **16.** Ex. 8. **18.** Ex. 10.

192. Two Equations Homogeneous except for Absolute Terms. If two quadratic equations are homogeneous except for the absolute terms, they can always be solved. For we may eliminate the absolute term from either equation and apply § 191.

Solve the equations

$$x^2 + xy + 2y^2 = 44 \quad (1)$$

$$2x^2 - xy + y^2 = 16 \quad (2)$$

Multiplying (1) by 4, $4x^2 + 4xy + 8y^2 = 176.$ (3)

Multiplying (2) by 11, $22x^2 - 11xy + 11y^2 = 176.$ (4)

Dividing (4) − (3) by 3, $6x^2 - 5xy + y^2 = 0.$

Factoring, $(3x - y)(2x - y) = 0.$

From $3x - y = 0$ we have $y = 3x.$

From $2x - y = 0$ we have $y = 2x.$

Substituting $3x$ for y in (2),

$2x^2 - 3x^2 + 9x^2 = 16.$

$8x^2 = 16.$

$x^2 = 2.$

$x = \pm\sqrt{2}.$

$y = 3x = \pm 3\sqrt{2}.$

Substituting $2x$ for y in (2),

$2x^2 - 2x^2 + 4x^2 = 16.$

$4x^2 = 16.$

$x^2 = 4.$

$x = \pm 2.$

$y = 2x = \pm 4.$

Therefore, when $x = +\sqrt{2}, \ -\sqrt{2}, \ +2, \ -2,$

the corresponding value of $y = +3\sqrt{2}, \ -3\sqrt{2}, \ +4, \ -4.$

Exercise 93. Simultaneous Quadratics

Solve the equations:

1. $x^2 + xy + 2y^2 = 11$
   $2x^2 - xy + 2y^2 = 8$

2. $x^2 - xy + y^2 = 7$
   $3x^2 + 13xy + 8y^2 = 162$

3. $2x^2 + 3xy + y^2 = 12$
   $6x^2 + 6xy - y^2 = 14$

4. $x^2 - xy - y^2 = 5$
   $2x^2 + 3xy + y^2 = 28$

5. $x^2 + 3xy + y^2 = 1$
   $3x^2 + xy + 3y^2 = 13$

6. $x^2 - 2xy + 3y^2 = 1\frac{2}{9}$
   $x^2 + xy - y^2 = \frac{1}{9}$

7. $x^2 + 3xy - y^2 = 3$
   $2x^2 + 5xy + y^2 = 8$

8. $2x^2 + 3xy + 4y^2 = 36$
   $3x^2 - 2xy - 3y^2 = -8$

193. Special Forms of Quadratics. The two cases mentioned on pages 199–201 are the only ones likely to be met that can always be solved by quadratic methods. Special forms will arise that can be so solved, sometimes involving equations of degree higher than the second, and sometimes involving linear equations. Such cases usually require only a little ingenuity in treatment.

1. Solve the equations

$$x^3 + y^3 = 9 \qquad (1)$$

$$x + y = 3 \qquad (2)$$

Dividing (1) by (2), $x^2 - xy + y^2 = 3.$ (3)

From (2), $y = 3 - x.$

Substituting $3 - x$ for y in (3), $3x^2 - 9x + 6 = 0,$

or $x^2 - 3x + 2 = 0,$

or $(x - 2)(x - 1) = 0.$

Therefore $x = 2$ or $1,$

$y = 1$ or $2.$

Check. $2^3 + 1^3 = 9$, and $2 + 1 = 3;$

$1^3 + 2^3 = 9$, and $1 + 2 = 3.$

2. Solve the equations

$$x^2 + y^2 + x + y = 18 \qquad (1)$$

$$xy = 6 \qquad (2)$$

From (2), $2xy = 12.$

Adding this to (1), $x^2 + 2xy + y^2 + x + y = 30,$

or $(x + y)^2 + (x + y) - 30 = 0.$

Factoring, $[(x + y) + 6][(x + y) - 5] = 0.$

Therefore $x + y = -6,$ and $y = -6 - x,$

or $x + y = 5,$ and $y = 5 - x.$

Substituting these values of y in (2),

$-6x - x^2 = 6.$ $\qquad$ $5x - x^2 = 6.$

$x^2 + 6x + 6 = 0.$ $\qquad$ $x^2 - 5x + 6 = 0.$

$\therefore x = -3 \pm \sqrt{3}.$ $\qquad$ $\therefore x = 2$ or $3.$

$\therefore y = -3 \mp \sqrt{3}.$ $\qquad$ $\therefore y = 3$ or $2.$

We may check by substituting in equation (1).

3. Solve the equations

$$\frac{1}{x}+\frac{1}{y}=\frac{3}{2} \tag{1}$$

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4} \tag{2}$$

From (1), $\frac{1}{y}=\frac{3}{2}-\frac{1}{x}$.

Substituting for $\frac{1}{y}$ in (2), $\frac{1}{x^2}+\frac{9}{4}-\frac{3}{x}+\frac{1}{x^2}=\frac{5}{4}$,

whence $\frac{2}{x^2}-\frac{3}{x}+1=0$.

Solving, $x=2$ or 1.

Substituting 2 for x, $\frac{1}{y}=\frac{3}{2}-\frac{1}{x}=\frac{3}{2}-\frac{1}{2}=1$,

whence $y=1$.

Substituting 1 for x, $\frac{1}{y}=\frac{3}{2}-1=\frac{1}{2}$,

whence $y=2$.

Therefore $x=2$ when $y=1$, and $x=1$ when $y=2$.

194. Symmetric Equations. When two unknown quantities can be interchanged without altering an equation, the equation is said to be *symmetric* with respect to the two quantities.

For example, in the equation $x+y=5$ we may interchange x and y without altering the equation, since $y+x=5$ is evidently the same as $x+y=5$. But the equation $x+2y=5$ is not symmetric, for it is not the same as $y+2x=5$. The equations in Exs. 1–3 above are all symmetric. Two symmetric *quadratic* equations can always be solved by quadratic methods.

195. Roots of Symmetric Equations. *In a set of equations symmetric with respect to x and y the values of x and y are always the same, although differently arranged.*

Since y may be put in place of x, in such cases we must get the same values for y as for x. It will be seen that this is so in Exs. 1–3 above. Therefore, having obtained the values of x, we may write down the values of y without solving, simply arranging them in proper pairs so that they correspond and may be checked.

Exercise 94. Simultaneous Quadratics

Solve the following equations:

1. $\dfrac{x+y}{x-y}=12$
   $x^2-y^2=48$

2. $16xy-x^2y^2=60$
   $x+y=7$

3. $x^2-y^2=228$
   $xy-y^2=42$

4. $x=\frac{7}{3}\sqrt{x+y}$
   $y=\frac{14}{3}\sqrt{x+y}$

5. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{9}{20}$
   $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{41}{400}$

6. $x^2+y^2-91=xy$
   $x^2+y^2+xy=223$

7. $42(x^2+y^2)=85xy$
   $15xy=2520$

8. $x^2+y^2+xy=67$
   $x^2+y^2=53$

9. $x^3+y^3=65$
   $x+y=5$

10. $x^3+y^3=1304$
    $x+y=8$

11. $x^2+xy=77$
    $xy+y^2=44$

12. $x^2+xy=616$
    $y^2+xy=1320$

13. $\dfrac{1}{x}-\dfrac{1}{y}=\dfrac{1}{36}$
    $xy^2-x^2y=324$

14. $x^2+xy=260$
    $xy+y^2=140$

15. $xy+x=20$
    $xy-y=12$

16. $x=6\sqrt{x+y}$
    $y=2\sqrt{x+y}$

17. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}$
    $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{5}{36}$

18. $x^2-xy=45$
    $xy-y^2=-36$

19. $x^2+xy=55$
    $y^2+xy=66$

20. $x+xy+y=5$
    $x^2+xy+y^2=7$

21. $x^3+y^3=37$
    $x^2-xy+y^2=37$

22. $x^2+y=5(x-y)$
    $x+y^2=2(x-y)$

23. $x+\sqrt{xy}+y=14$
    $x^2+xy+y^2=84$

24. $x+\sqrt{xy}+y=21$
    $x^2+xy+y^2=189$

Exercise 95. Problems involving Quadratics

All Problems for Written Work. Reject Meaningless Results

1. The lengths of the three sides of a right triangle are consecutive numbers. What are the lengths?

2. Both the sum and the product of two numbers are 10. Find the numbers.

3. The sum of two numbers is 3.2 and the sum of their squares is 5.62. Find the numbers.

4. Two poles standing on level ground 30 ft. apart are respectively 50 ft. and 75 ft. high. How far from the ground must the higher one break so that the top, as it bends over, may just reach the top of the other?

5. A gravel walk is to be made around two sides of a rectangular plot of ground, inside the boundary. The plot is 80 ft. wide and 100 ft. long. How wide must the walk be so that it shall cover half the plot?

6. If the length and breadth of a rectangle were each increased by 1, the area would be 48, and if they were each diminished by 1, the area would be 24. Find the dimensions.

7. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

8. The difference of the squares, the sum, and the product of two numbers are all equal. Find the numbers.

9. The difference of two numbers is three eighths of the greater, and the sum of their squares is 356. Find the numbers.

10. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$. If the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

11. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{5}{18}$ of their difference, and the difference of their squares is 3960. Find the numbers.

12. The fore wheel of a carriage turns in a mile 132 times more than the rear one. If the circumferences were each increased by 2 ft., it would turn only 88 times more. Find the circumference of each.

13. A number is formed by two digits. The second digit is less by 8 than the square of the first digit. If nine times the first digit is added to the number, the order of the digits is reversed. Find the number.

14. The numerator and denominator of a certain fraction are each greater by 1 than those of a second fraction, and the sum of the two fractions is $\frac{4}{7}\frac{3}{2}$. If the numerators were interchanged, the sum of the two fractions would be $\frac{5}{6}$. Find the fractions.

15. Three students, A, B, and C, agree to work out a set of problems in preparation for an examination, each to solve every problem. A solves nine problems a day and finishes the set 4 da. before B. B solves two more problems a day than C and finishes 6 da. before him. Find the number of problems in the set.

16. A and B have a certain manuscript to copy between them. At A's rate of work he would copy the whole manuscript in 18 hr., and B copies nine pages an hour. A finishes his portion in as many hours as he copies pages per hour. B takes 2 hr. longer than A in doing his share of the work. Find the number of pages copied by each.

17. A boat's crew, rowing at half their usual speed, row 3 mi. downstream and back again, accomplishing the distance in 2 hr. 40 min. At full speed they can go over the same course in 1 hr. 4 min. Find the rate of their rowing and the rate of the current.

18. A railroad train, after traveling 1 hr. from A, meets with an accident that delays it 1 hr. It then proceeds at a rate 8 mi. per hour less than its former rate, and reaches B 5 hr. late. If the accident had happened 50 mi. farther on, the train would have been only 3 hr. 20 min. late. How far is it from A to B?

CHAPTER X

RATIO, PROPORTION, AND VARIATION

196. Ratio. The relation of one number to another number of the same kind, as indicated by division, is called the *ratio* of the first to the second.

The ratio of $a + b$ to $c + d$ is indicated thus: $\frac{a+b}{c+d}$ or $a + b : c + d$.

Hence all ratios may be looked upon as fractions.

197. Terms of a Ratio. The two numbers involved in a ratio are called the *terms* of the ratio. The dividend is called the *antecedent*, and the divisor is called the *consequent*.

Thus we have

$$\frac{a}{b} = \frac{\text{antecedent}}{\text{consequent}} = \frac{\text{numerator}}{\text{denominator}} = \frac{\text{dividend}}{\text{divisor}}.$$

If $\frac{a}{b} = \frac{2}{3}$, we say that a is to b as 2 is to 3.

198. Greater and Less Inequality. If the absolute value of a ratio is greater than 1, the ratio is called a *ratio of greater inequality*. If the absolute value of a ratio is less than 1, the ratio is called a *ratio of less inequality*.

Thus $\frac{3}{2}$ is a ratio of greater inequality, and $\frac{2}{3}$ is a ratio of less inequality. The symbol $>$ is read "is greater than," and the symbol $<$ is read "is less than."

199. Laws of Ratio. The following are the important laws of a ratio whose terms are positive.

1. *Both terms of a ratio may be multiplied, or both may be divided, by the same number without changing the value of the ratio.*

A ratio being a fraction, the laws of fractions are true for ratios.

2. *A ratio of greater inequality is decreased, and a ratio of less inequality is increased, by adding the same positive number to both terms.*

Let a be the antecedent and b the consequent.

Then $\quad \frac{a+n}{b+n} > \text{ or } < \frac{a}{b}$,

according as $\quad ab + bn > \text{ or } < ab + an$,

or as $\quad bn > \text{ or } < an$,

or as $\quad b > \text{ or } < a$.

That is, if $\quad a > b$, then $\frac{a+n}{b+n} < \frac{a}{b}$,

and if $\quad a < b$, then $\frac{a+n}{b+n} > \frac{a}{b}$.

3. *A ratio of greater inequality is increased, and a ratio of less inequality is decreased, by subtracting the same positive number from both terms.*

We must first notice that

$$-4 > -5, \text{ but } 4 < 5,$$

and that, in general, if $\quad -b > -a$, then $b < a$.

That is, changing the sign changes the order of inequality.

We see that $\quad \frac{a-n}{b-n} > \text{ or } < \frac{a}{b}$,

according as $\quad ab - bn > \text{ or } < ab - an$,

or as $\quad -bn > \text{ or } < -an$,

or as $\quad b < \text{ or } > a$.

4. *In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Consider the three ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, and let each equal r.

Since $\quad \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r, \ \therefore a = br, c = dr, \text{ and } e = fr.$

Adding, $\quad a + c + e = (b + d + f)r.$

Dividing, $\quad \frac{a+c+e}{b+d+f} = r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$

Exercise 96. Ratios

Examples 1 to 4, oral — Examples 5 to 17, written

1. Express in simplest fractional form the ratio of 20 to 40; of 30 to 40; of 35 to 50; of 48 to 50; of 25 to 35.

2. Express in simplest form, as improper fractions, the ratio of 25 to 10; of 30 to 12; of 33 to 21; of 80 to 60.

3. How is the ratio of 7 to 4 changed by multiplying both terms by 3? by adding 3 to both terms? by subtracting 3 from both terms?

4. How is the ratio of 8 to 10 changed by multiplying both terms by 2? by dividing both terms by 2? by adding 2 to both terms? by subtracting 2 from both terms?

Simplify the following ratios:

5. $a^4 - b^4 : a^6 - b^6$.

6. $\dfrac{a}{p+q} : \dfrac{a}{p^2+2pq+q^2}$.

7. $x^3 - 1 : x^2 + x + 1$.

8. $3 + \dfrac{m}{n} : 3 - \dfrac{m}{n}$.

9. Separate 100 into two parts having the ratio 3 : 7.

Let x = the smaller part.

Then $100 - x$ = the larger part.

Then $\dfrac{x}{100-x} = \dfrac{3}{7}$.

Therefore $x = 30$, and $100 - x = 70$.

Separate into two parts having the ratios specified:

10. 36, 2 : 1.

11. 45, 6 : 3.

12. 49, 1 : 6.

13. 208, 3 : 5.

14. 320, 9 : 7.

15. 612, 7 : 11.

16. Two cogwheels are geared together. The distance between their centers is 40 in. What are their diameters if their speeds have a ratio of 8 to 10?

17. A wheel 60 in. in diameter, making 150 revolutions per minute, is belted to another wheel 30 in. in diameter. Find the speed of the smaller wheel.

200. Proportion. An expressed equality of two ratios is called a *proportion.*

Thus if $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d, are said to be in proportion or to form a proportion.

This proportion is often read, "a is to b as c is to d," and may be written thus: $a : b = c : d$. In older works it is often written $a : b : : c : d$.

201. Terms of a Proportion. In a proportion the first and fourth terms are called the *extremes,* and the second and third terms the *means.*

In the proportion $a : b = c : d$ the extremes are a and d, and the means are b and c.

In the same proportion d is called the *fourth proportional* to a, b, and c.

202. Mean Proportional. If the means of a proportion are the same, this term is called the *mean proportional* between the other two.

Thus m is the mean proportional between x and y if $x : m = m : y$. In this proportion y is called the *third proportional* to x and m.

203. Laws of Proportion. The following are the important laws of proportion:

1. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $\frac{a}{b} = \frac{c}{d}$.

Multiplying by bd, $ad = bc$.

2. *If the product of two numbers equals the product of two other numbers, either two may be made the means of a proportion, and the other two the extremes.*

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$,

or $\frac{a}{b} = \frac{c}{d}$.

3. *If* $a : b = c : d$, *then* $b : a = d : c$.

In this case the first proportion is said to be taken by *inversion*.

From Law 1, $bc = ad$.

Dividing by ac, $\frac{bc}{ac} = \frac{ad}{ac}$,

or $\frac{b}{a} = \frac{d}{c}$.

4. *If* $a : b = c : d$, *then* $a : c = b : d$.

In this case the first proportion is said to be taken by *alternation*.

From Law 1, $ad = bc$.

Dividing by cd, $\frac{ad}{cd} = \frac{bc}{cd}$,

or $\frac{a}{c} = \frac{b}{d}$.

5. *If* $a : b = c : d$, *then* $a + b : b = c + d : d$.

In this case the first proportion is said to be taken by *composition*.

Given $\frac{a}{b} = \frac{c}{d}$.

Adding 1, $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or $\frac{a+b}{b} = \frac{c+d}{d}$.

6. *If* $a : b = c : d$, *then* $a - b : b = c - d : d$.

In this case the first proportion is said to be taken by *division*.

Given $\frac{a}{b} = \frac{c}{d}$.

Subtracting 1, $\frac{a}{b} - 1 = \frac{c}{d} - 1$,

or $\frac{a-b}{b} = \frac{c-d}{d}$.

7. *The product of the means divided by either extreme equals the other extreme. The product of the extremes divided by either mean equals the other mean.*

The proof is left for the student.

Exercise 97. Proportion

Examples 1 to 6, oral — Examples 7 to 39, written

1. Solve the equation $\frac{1}{4}x = \frac{5}{8}$.

2. In the proportion $\frac{x}{4} = \frac{5}{8}$, find the value of x.

3. In the proportion $x : 4 = 5 : 8$, find the value of x.

Find the value of x:

4. $\frac{x}{4} = \frac{7}{2}$.

5. $\frac{x}{5} = \frac{9}{25}$.

6. $\frac{x}{7} = \frac{9}{35}$.

7. $\frac{5}{12} = \frac{x}{9}$.

8. $\frac{3}{8} = \frac{8}{x}$.

9. $\frac{5}{x} = \frac{25}{26}$.

10. $\frac{7}{x} = \frac{42}{75}$.

11. $\frac{39}{40} = \frac{3}{x}$.

12. $\frac{75}{83} = \frac{5}{x}$.

13. If the means are 3 and 15, and one extreme is 5, what is the other extreme?

14. If $x - 5 : 5 = 56 : 20$, by what law do we know that $x : 5 = 76 : 20$? What is the value of x?

Find the value of x:

15. $x : 2.4 = 2.42 : 2.2$.

16. $x - 32 : 32 = 10 : 2$.

17. $2.6 : x = 2.2 : 15.4$.

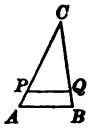
18. $17 - 3x : 3x = 7 : 8$.

19. If each mean of a proportion is 16, and one extreme is 64, what is the other extreme?

20. In a lever PW, if sufficient power (p) is applied at P, a weight (w) at W can be lifted. If F is the fulcrum, it is known that $p : w$ equals the ratio $FW : FP$. If $FW = 20$ in., $FP = 60$ in., and $w = 480$ lb., what does p equal?

21. If two boys weigh respectively 90 lb. and 108 lb., where must the fulcrum be placed under a 10-foot board so that the boys, sitting at the ends, will just balance? (Use the law given in Ex. 20.)

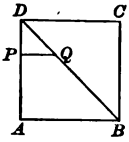
22. It is proved in geometry that a line parallel to the base of a triangle divides the other two sides proportionally. In this figure, if $AP = 3$, $PC = 12$, and $QC = 10$, find the length of BQ.



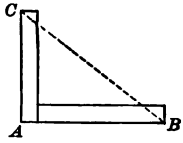
23. If a line parallel to the base of a triangle divides one side into the parts 15 and $8\frac{1}{2}$, and the other side into the corresponding parts 6 and x, what is the value of x?

24. If two sides of a triangle are 14 and 18, and a line parallel to the base cuts the former into the parts 6 and 8, into what lengths will it cut the latter?

25. In the square here shown, PQ is $\parallel$ to AB. If a side of the square is 20 in., $DB = 28.28$ in. If $DP = 6$ in., what is the length of DQ?

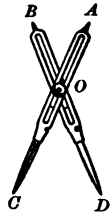


26. Two pieces of timber 1 ft. wide are fitted together at right angles as here shown. AB is 16 ft. long, AC is 12 ft. long, and the distance BC along the dotted line is 20 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece; across the horizontal piece. $x : 20 = 1 : 12$ is one proportion.



27. It is proved in geometry that two triangles of the same shape have their corresponding sides proportional. Such triangles are said to be *similar*. In Ex. 22 the triangles ABC and PQC are similar. If $AB = 4$, $AC = 3\frac{3}{4}$, and $PC = 3$, what is the length of PQ?

28. The figure at the right represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O, the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. The triangle formed by O, A, and B is always similar to the triangle formed by O, C, and D. If $OA = 1\frac{1}{2}$ in. and $OC = 2\frac{1}{2}$ in., then AB is what part of CD?



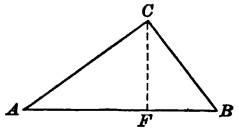
29. The lengths of two rectangular fields are in the ratio of 2 to 3, and the breadths in the ratio of 5 to 6. Find the ratio of their areas.

30. Two men are paid in proportion to the work they do. M can do in 20 da. the work that it takes N 24 da. to do. Compare their wages.

31. If five gold coins and four silver coins are worth as much as three gold coins and twelve silver coins, find the ratio of the value of a gold coin to that of a silver coin.

32. A line is divided into two parts in the ratio of 2 to 3, and into two parts in the ratio of 3 to 4. The distance between the points of section is 2 in. Find the length of the line.

33. If a perpendicular is let fall from the vertex of the right angle upon the hypotenuse of a right triangle, it divides the right triangle into two triangles similar to the original triangle and to each other. In the given figure, if $AF = 15$ and $CF = 10.5$, what is the length of FB?



34. In the same figure, if $AB = 14.9$ and $AC = 12.2$, what is the length of AF?

35. If a boy $4\frac{3}{4}$ ft. tall casts a shadow $4\frac{1}{2}$ ft. long at the same time that the church spire casts a shadow 135 ft. long, how high is the church spire?

36. If a boy, lying down with his eye to the ground, sights over the top of a 5-foot pole held vertically $3\frac{1}{8}$ ft. from his eye, and can just see the top of a tree $37\frac{1}{2}$ ft. from his eye, how tall is the tree?

37. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. This is called the taking of the proportion by *composition and division.*

38. Solve the equation $\frac{x+3}{x-3} = \frac{2}{1}$ by the principle of Ex. 37.

39. Solve the equation $\frac{x-15}{x+15} = \frac{1}{7}$ by the principle of Ex. 37.

204. Variation. If two variable quantities, x and y, have a constant ratio k, either quantity is said to *vary* as the other.

If $\frac{x}{y} = k$, then $x = ky$, and $y = \frac{1}{k} \cdot x$.

The expression "x varies as y" is sometimes written $x \propto y$, which means that $x = ky$, k being the constant ratio.

For example, in a circle $c = 2\pi r$, so that $c \propto r$, c always being 2π times r.

Similarly, for the area of a circle, $a = \pi r^2$, in which case $a \propto r^2$.

The subject of variation required in certain courses of study in connection with ratio may be omitted without affecting the subsequent work.

If $x \propto y$, and $x = 12$ when $y = 4$, find x when $y = 6$.

Since $x \propto y$, therefore $x = ky$, where k is some constant.

Hence $12 = k \cdot 4$,

and therefore $3 = k$.

Hence, when $y = 6$,

$x = k \cdot 6 = 3 \cdot 6 = 18$.

Exercise 98. Variation

Examples 1 to 3, oral — Examples 4 to 11, written

1. If $x = ky$, find k when $x = 21$ and $y = 10.5$.
2. If $x = ky$, find y when $x = 27$ and $k = 4.5$.
3. If $a = \pi r^2$, find a when $\pi = 3.1416$ and $r = 100$.
4. If $x \propto y$, and $x = 19$ when $y = 7$, find x when $y = 10.5$.
5. If $x \propto y$, and $x = 17$ when $y = 9$, find y when $x = 25.5$.
6. If $x \propto y$, and the ratio of x to y is $\frac{3}{4}$, find x when $y = 28$.
7. In Ex. 6 find y when $x = 36$; when $x = 4.83$.
8. If $x = ky$, and $y = mz$, find x in terms of z.
9. If $x \propto y$, and $y \propto z$, show that $x \propto z$.
10. If $x \propto y$, and $x \propto y$, show that $y \propto z$.
11. If the circumference of a circle varies as the radius, and the radius varies as the diameter, show that the circumference varies as the diameter.

205. Joint Variation. The area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant. If both base and altitude vary, the area varies as their product.

In other words, if the base remains unchanged, the area will be doubled by doubling the altitude. Likewise, if the altitude remains unchanged, the area will be doubled by doubling the base. But if we double both base and altitude, the area will be multiplied by 4.

That is, $a \propto bh$, or $a = kbh$. In fact, as we know, $a = \frac{1}{2}bh$, so that in this case $k = \frac{1}{2}$. In general, if $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when both change, as shown in § 207.

206. Inverse Variation. If two variables, x and y, are so related that the ratio of x to $\frac{1}{y}$ is some constant number, k, either quantity is said to *vary inversely* as the other.

If $x : \frac{1}{y} = k$, then $xy = k$, or $x = \frac{k}{y}$, and $y = \frac{k}{x}$.

As y increases, x decreases. If y is made twice as large, x becomes half as large.

For example, if a certain number of men, m, can do a piece of work in t days, then twice as many men can do the work in $\frac{1}{2}t$ days, and $\frac{1}{4}$ as many men can do it in $4t$ days. That is, $m \propto \frac{1}{t}$.

1. If $x \propto \frac{1}{y}$, and $x = 9$ when $y = 7$, find x when $y = 2$.

Since $x \propto \frac{1}{y}$, we have $xy = k$.

Hence $9 \cdot 7 = 63 = k$.

Hence $x \cdot 2 = 63$, when $y = 2$,

and $x = 31.5$.

2. If x varies directly as y and inversely as z, and if $x = 12$ when $y = 6$ and $z = 2$, find x when $y = 8$ and $z = 1$.

Since $x \propto y$, and $x \propto \frac{1}{z}$, then $x = ky \cdot \frac{1}{z}$ (§ 205).

Hence $12 = k \cdot 6 \cdot \frac{1}{2}$,

and $k = 4$.

Then $x = 4 \cdot 8 \cdot 1 = 32$,

when $y = 8$ and $z = 1$.

Exercise 99. Problems in Variation

All Examples written

1. If x varies inversely as y, and $x = 12$ when $y = 2$, find x when $y = 7$.

2. If $x \propto y$, and $x \propto \frac{1}{z}$, and if $x = 6$ when $y = 8$ and $z = 12$, find x when $z = 28$ and $y = 4$.

3. If $x \propto y$, and $x \propto \frac{1}{z}$, and if $x = 15$ when $y = 3$ and $z = 5$, find y when $x = 35$ and $z = 12.5$.

4. If $y \propto \frac{1}{x}$, and $y \propto \frac{1}{z}$, and if $x = 2$ when $y = 4$ and $z = 3$, find z when $x = 4$ and $y = 1$.

5. If $xy = a$, and $yz = b$, and if $x = 4$ when $y = 8$ and $z = 6$, find z when $x = 8$ and $y = 2$.

6. The weight of a sphere of steel varies as the volume, and the volume varies as the cube of the diameter. If a sphere 1 in. in diameter weighs 0.15 lb., find the weight of a sphere 4 in. in diameter.

Representing the weight, volume, and diameter by w, v, and d respectively, we have $w \propto v$, and $v \propto d^3$.

Hence $w = mv$, and $v = nd^3$,

so that $w = mnd^3$;

or, we may say, $w = kd^3$.

Therefore $0.15 = k \cdot 1^3 = k$.

Hence, when $d = 4$, $w = 0.15 \cdot 4^3 = 9.6$.

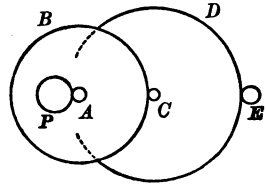
Hence a sphere of steel 4 in. in diameter weighs 9.6 lb.

7. The diagonal of a cube varies as the edge. When the edge is 6 in. the diagonal is 10.4 in. Find the diagonal when the edge is 15 in.

8. The capacity of a water main varies as its length and also as the square of its diameter. If a pipe 1 ft. in diameter and 16 ft. long has a capacity of 12.576 cu. ft., what is the capacity of a water main 10 in. in diameter and 480 ft. long?

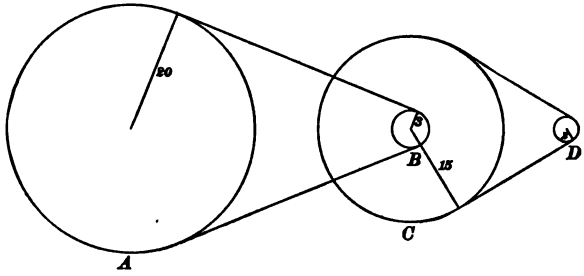
9. The value of a diamond varies as the square of the weight. If a diamond worth \$640 is cut into two pieces whose weights are as 1 to 3, what is the value of each piece?

10. The figure at the right represents several wheels geared together. P has 50 teeth, A 20 teeth, B 200 teeth, C 15 teeth, D 250 teeth, and E 25 teeth. If P makes 25 revolutions per minute, how many revolutions per minute will E make?



11. A cube of clay 14 in. on an edge is molded into a right prism whose base is a square 7 in. on a side. Find the height of the prism.

12. The intensity of light varies inversely as the square of the distance. How far from a lamp is a point that receives half as much light as another point 16 ft. from the lamp?



13. Several wheels are belted together as here shown. The circumferences of circles varying as the radii, and the radii being as indicated, how many revolutions per minute will D make when A is making 2000 revolutions per minute?

14. The volume of a right circular cylinder varies as the product of its height and the square of its radius. A cylindric can is 5 in. high and 1.5 in. in radius, and another is 6 in. high and has a radius of 2 in. Find the radius of a third can 7 in. high that will hold as much as the other two together.

207. Theory of Variation. Certain practical cases of variation have been considered without strict proof. The theory may be summed up, for our purposes, in three laws:

1. If $y \propto x$, and $x \propto z$, then $y \propto z$.

For $y = mx$ and $x = nz$,

hence $y = mnz$,

and therefore $y \propto z$.

2. If $y \propto x$, and $x \propto z$, then $y \pm z \propto x$.

For $y = mx$ and $z = nx$,

hence $y \pm z = (m \pm n)x$,

and therefore $y \pm z \propto x$.

3. If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when x and z are both variables.

Let x', z', and x'', z'', be two sets of corresponding values of the variables x, z, and let y' correspond to x' and y'' to z''.

Let x change from x' to x'', z remaining constant, and let Y be the value of y corresponding to x''.

Then $y' : Y = x' : x''$.

Now let z change from z' to z'', x remaining constant.

Then $Y : y'' = z' : z''$.

Then, by multiplying, $y' : y'' = x'z' : x''z''$,

or $y' : x'z' = y'' : x''z''$ (§ 203).

That is, the ratio $y : xz$ is constant, and hence $y \propto xz$.

For example, the area (y) of a rectangle varies as the base (x) when the altitude (z) is constant, and varies as the altitude (z) when the base (x) is constant, but it varies as the product (xz) of the base and altitude when both vary.

In the same way, if y varies as each of the quantities u, v, w, $\cdots$, when the rest are unchanged, then when all change, $y \propto uvw \cdots$.

It will be seen by the above that the theory of variation is merely a special case of the theory of ratio. Indeed, since the symbols $x \propto y$ mean merely that $x = ky$, or $\frac{x}{y} = k$, there is no real need for having a special symbol for variation.

208. Illustrative Problem. The volume (v) of a certain solid on a circular base varies as the square of the radius (r) when the height (h) is constant, and as the height when the base is constant. When $r = 3\frac{1}{2}$ in. and $h = 7\frac{1}{2}$ in., then $v = 96\frac{1}{4}$ cu. in. Find the value of h when $v = 16\frac{1}{2}$ cu. in. and $r = 1\frac{1}{2}$ in.

Since $$96\tfrac{1}{4} = k(3\tfrac{1}{2})^2 \cdot 7\tfrac{1}{2},$$

we have $$\frac{385}{4} = k \cdot \frac{7 \cdot 7 \cdot 15}{2 \cdot 2 \cdot 2},$$

whence $$k = \tfrac{2}{3}\tfrac{2}{1}.$$

Then $$16\tfrac{1}{2} = \tfrac{2}{3}\tfrac{2}{1}(1\tfrac{1}{2})^2 \cdot h,$$

and therefore $$h = 7.$$

Exercise 100. Problems

1. The velocity acquired by a stone falling from rest varies as the time of falling, and the distance fallen varies as the square of the time. In 3 sec. a stone will fall 145 ft. and acquire a velocity of $96\frac{2}{3}$ ft. per sec. Find the velocity and distance at the end of 5 sec.

2. If a weight draws up a lighter weight by means of a string passing over a fixed pulley, the space described in a given time will vary directly as the difference between the weights and inversely as their sum. If 9 oz. draw 7 oz. through 8 ft. in 2 sec., through what distance will 12 oz. draw 9 oz. in the same time?

3. The space will also vary as the square of the time. Find the space in Ex. 2 if the time in the latter case is 3 sec.

4. Equal volumes of iron and copper weigh 77 oz. and 89 oz. respectively. Find the weight of 10.5 ft. of round copper rod when 9 in. of iron rod of the same diameter weigh 31.9 oz.

5. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the earth and Mercury from the sun being 91 and 35 millions of miles respectively, find the number of days in Mercury's revolution.

CHAPTER XI

PROGRESSIONS

209. Series. A succession of terms that proceed according to some fixed law is called a *series.*

For example, the natural numbers, 1, 2, 3, $\cdots$, form a series, the law being that each term is one more than the preceding term.

The study of series is an important part of higher mathematics. For example, by means of series we find the approximate value of π by an easier method than the one given in geometry.

210. Finite Series. A series in which the number of terms is limited is called a *finite series.*

If the number of terms of a series is unlimited, it is called an *infinite series.*

For example, 2, 6, 18, is a finite series, having only three terms. The fixed law is that each term after the first is three times the preceding term of the series.

The series 1, 2, 3, $\cdots$, and so on forever is an infinite series, as is also the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\cdots$.

The number of different kinds of series is evidently unlimited. For example, 1^1, 2^2, 3^3, 4^4, $\cdots$, is a series, and so are 1, -2, $+3$, -4, $+5$, -6, $\cdots$, and 3^{-1}, 4^{+2}, 5^{-3}, 6^{+4}, $\cdots$.

Only two kinds of series are commonly considered in elementary algebra, the arithmetical and the geometric.

211. Arithmetical Progression. A finite series in which each succeeding term after the first may be found by adding a constant quantity to the preceding term is called an *arithmetical progression.*

The words *series* and *progression* are generally used interchangeably in elementary algebra.

For example, 2, 4, 6, 8, 10, is an arithmetical series or an arithmetical progression.

212. Elements of an Arithmetical Progression. An arithmetical progression may be represented by

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \cdots,$$

in which a is the first term and d is the constant quantity added.

It is customary to speak of d as the *common difference*, and to represent the various elements as follows:

a = first term, $\quad d$ = common difference,

l = last term, $\quad n$ = number of terms,

and to let s stand for the sum of all the terms.

If d is positive, the series is an *increasing series*, as in the case of 2, 5, 8, 11, where $a = 2$, $d = 3$, $n = 4$, $l = 11$.

If d is negative, the series is a *decreasing series*, as in the case of 16, 12, 8, 4, 0, -4, in which $a = 16$, $d = -4$, $n = 6$, $l = -4$.

The terms between the first and last terms are called *arithmetical means*.

213. The *n*th Term, or *l*. Since each succeeding term of an arithmetical progression is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term. Hence the coefficient of d in the nth term is $(n-1)$. Calling the nth term l, we have

$$l = a + (n-1)d.$$

Thus in the series 2, 5, 8, 11, we see that the last term is 11, and that $11 = 2 + (4-1)\cdot 3 = 2 + 3\cdot 3$.

214. The Sum of the Terms. Indicating the sum of the terms, we have

$$s = a + (a+d) + (a+2d) \quad + \cdots + (l-d) + l$$

or
$$s = l \quad + (l-d) + (l-2d) \quad + \cdots + (a+d) + a$$

$$\therefore 2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)$$

$$= n(a+l).$$

$$\therefore s = \frac{n}{2}(a+l).$$

That is, the formula for the sum is

$$s = \frac{n}{2}(a+l).$$

215. Problems in Arithmetical Progressions. In the two formulas in §§ 213 and 214, when the values of any three of the letters are known, the values of the others may be found.

1. Find the tenth term in the series 1, 7, 13, · · ·.

We have $a = 1$, $d = 6$, $n = 10$.

Hence $$l = a + (n-1)d = 1 + 9 \cdot 6 = 55.$$

2. Find the sum of the terms of the series 1, 7, 13, · · · to ten terms.

As in Ex. 1, $l = 55$.

Hence $$s = \frac{n}{2}(a + l) = \frac{10}{2}(1 + 55) = 5 \cdot 56 = 280.$$

3. Write the series of which the first term is 5, the last term 33, and the sum of the terms 152.

Since $l = a + (n-1)d$, $$33 = 5 + (n-1)d.$$

Since $s = \frac{n}{2}(a + l)$, $$152 = \frac{n}{2}(5 + 33) = 19n.$$

$$\therefore 8 = n.$$

$$\therefore 33 = 5 + (8-1)d,$$

or $$4 = d.$$

Therefore the series is 5, 9, 13, 17, 21, 25, 29, 33.

4. Find a when d, l, and s are given.

From $l = a + (n-1)d$ we have $n = \dfrac{l - a + d}{d}$.

From $s = \frac{n}{2}(a + l)$ we have $n = \dfrac{2s}{a + l}$.

Therefore $$\frac{l - a + d}{d} = \frac{2s}{a + l}.$$

Simplifying, $$a^2 - ad = l^2 + ld - 2ds.$$

Solving for a, $$a = \tfrac{1}{2}\left[d \pm \sqrt{(2l + d)^2 - 8ds}\right].$$

216. Formulas of Arithmetical Progressions. Only the two fundamental formulas of §§ 213 and 214 need to be memorized. From them the formulas on page 224 may be deduced.

Teachers will use their discretion as to the amount of this work to be required.

No.	Given	Required	Result
1	$a\ d\ n$	l	$l = a + (n-1)d$
2	$a\ d\ s$	l	$l = \frac{1}{2}\left[-d \pm \sqrt{8ds + (2a-d)^2}\right]$
3	$a\ n\ s$	l	$l = \frac{2s}{n} - a$
4	$d\ n\ s$	l	$l = \frac{s}{n} + \frac{(n-1)d}{2}$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d]$
6	$a\ d\ l$	s	$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$
7	$a\ n\ l$	s	$s = \frac{n}{2}(a + l)$
8	$d\ n\ l$	s	$s = \frac{1}{2}n[2l - (n-1)d]$
9	$d\ n\ l$	a	$a = l - (n-1)d$
10	$d\ n\ s$	a	$a = \frac{s}{n} - \frac{(n-1)d}{2}$
11	$d\ l\ s$	a	$a = \frac{1}{2}\left[d \pm \sqrt{(2l+d)^2 - 8ds}\right]$
12	$n\ l\ s$	a	$a = \frac{2s}{n} - l$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}$
14	$a\ n\ s$	d	$d = \frac{2(s - an)}{n(n-1)}$
15	$a\ l\ s$	d	$d = \frac{l^2 - a^2}{2s - l - a}$
16	$n\ l\ s$	d	$d = \frac{2(nl - s)}{n(n-1)}$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1$
18	$a\ d\ s$	n	$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}$
19	$a\ l\ s$	n	$n = \frac{2s}{l+a}$
20	$d\ l\ s$	n	$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}$

Exercise 101. Arithmetical Progression

All Examples written

1. Find the ninth and twelfth terms of 3, 7, 11, · · ·.

2. Find the tenth and twentieth terms of 5, 11, 17, · · ·.

3. Find the sixth and eighth terms of 40, 32, 24, · · ·.

4. Find the fifteenth term of 12, 3, − 6, · · ·.

5. Find the sum of the first nine terms of 7, 11, 15, · · ·; also of the first twelve terms.

6. Find the sum of the first eight terms of − 28, − 20, − 12, · · ·; also of the first three terms; also of the first five terms.

7. Find the sum of the first seven terms of − 6, − 3, 0, · · ·; also of the first five terms.

8. Find the sum of the first sixteen terms of $3\frac{1}{2}$, 7, $10\frac{1}{2}$, · · ·; also of the first twenty terms.

9. Given $a = 7$, $l = 42$, $n = 6$, find d and s.

10. Given $a = 65$, $n = 9$, $s = 333$, find d and l.

11. Given $a = -27$, $d = 12$, $l = 45$, find n and s.

12. Given $a = 7$, $l = 49$, $s = 812$, find d and n.

13. Given $d = \frac{2}{3}$, $n = 24$, $s = 56$, find a and l.

14. Given $a = 4$, $d = 3$, $s = 246$, find l and n.

15. Insert four terms in an arithmetical series between − 15 and 45, that is, four arithmetical means, thus making six terms in the series.

16. Insert seven arithmetical means between 3 and 51.

17. Insert twelve arithmetical means between 3 and 42.

18. Insert six arithmetical means between 1 and 5.

19. The first term of an arithmetical series is 21 and the third term is 33. Find the sum of five terms.

20. The first term of an arithmetical series is − 3, and the sum of the first five terms is 105. What term is 33?

21. The sum of three numbers in an arithmetical series is 120. The difference between the first and last terms is 26. Find the series.

22. What is the sum of the first hundred positive integers?

23. What is the sum of the first ten odd numbers? of the first twenty? of the first n?

24. What is the sum of the first ten numbers that are divisible by 5?

25. What is the sum of the first ten numbers beginning with 15 that are divisible by 3?

26. In a potato race 100 potatoes are placed 3 ft. apart in a straight line. A runner picks up one potato at a time and carries it to a basket in the line of the potatoes, and 3 ft. back of the first one. How far does he run?

27. In a potato race, if there are 50 potatoes 6 ft. apart, and the basket is 6 ft. back of the first one, how far does the contestant run?

28. A body falling freely falls 16.08 ft. in the first second, and in each succeeding second 32.16 ft. more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 12 sec., how far does it fall in the last second? How high is the balloon?

29. A baseball was dropped from the top of Washington Monument, 550 ft. high, and was caught by an American League catcher. How fast was the ball falling when caught?

30. Some railroads use 24-hour time, the hours being numbered from 1 to 24. If a clock should strike the hours on this plan, how many strokes would it strike in one day?

31. How many terms of the series 18, 15, 12, must be taken to have their sum 60? Write the series and explain the double answer.

32. How many terms of the series 40, 30, 20, must be taken to have their sum 90? Explain the two answers.

217. Geometric Progression. A finite series in which each succeeding term may be found by multiplying the preceding term by a constant multiplier is called a *geometric progression.*

The constant multiplier is called the *ratio.*

The first term is designated by a, the last term by l, the ratio by r, the number of terms by n, and the sum by s.

Special examples of a geometric progression are

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,$$

and

$$729,\ 243,\ 81,\ 27,\ 9,\ 3,\ 1,\ \tfrac{1}{3},$$

and the general form is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4, \cdots.$$

The terms between the first and last terms are called *geometric means.*

218. The *n*th Term, or *l*. Since each succeeding term of a geometric progression, after the first, is obtained by multiplying the preceding term by r, the exponent is always one less than the number of the term, so that the second term is ar, the third is ar^2, the tenth is ar^9, and the nth is ar^{n-1}. Hence

$$l = ar^{n-1}.$$

Thus in the series 5, 15, 45, 135, we see that $135 = 5 \cdot 3^3 = 5 \cdot 27$.

219. The Sum of the Terms. To find the sum, we have

$$s = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiplying by r, $\quad rs = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$

Subtracting, $\quad rs - s = ar^n - a, \quad$ or $\quad (r-1)s = ar^n - a.$

Hence

$$s = \frac{ar^n - a}{r - 1}.$$

Since $\quad l = ar^{n-1}, \quad$ therefore $lr = ar^n.$

$$\therefore\ s = \frac{lr - a}{r - 1}.$$

We therefore have two convenient formulas for s:

$$s = \frac{ar^n - a}{r - 1},$$

and

$$s = \frac{lr - a}{r - 1}.$$

220. Problems in Geometric Progressions. From the formulas in §§ 218 and 219, when the values of any three letters are known, the values of the other two may be found.

In the case of finding n, however, logarithms are needed except when n is easily determined by inspection. This case is therefore considered in only one simple example at this time.

1. The sum of the terms of a geometric progression is 381, the first term is 3, and the last term 192. Find the ratio and the number of terms.

Since $l = ar^{n-1}$,

therefore $192 = 3r^{n-1}$.

Since $s = \dfrac{lr - a}{r - 1}$,

therefore $381 = \dfrac{192r - 3}{r - 1}$,

from which $r = 2$.

Substituting, $192 = 3 \cdot 2^{n-1}$,

and $64 = 2^{n-1}$.

Since $64 = 2^6$, $n - 1 = 6$, and $n = 7$.

2. Given r, n, s, find l.

From $l = ar^{n-1}$, $a = \dfrac{l}{r^{n-1}}$.

Substituting in $s = \dfrac{lr - a}{r - 1}$, $s = \dfrac{rl - \dfrac{l}{r^{n-1}}}{r - 1}$

$$= \frac{l(r^n - 1)}{r^{n-1}(r - 1)}.$$

Solving for l, $l = \dfrac{sr^{n-1}(r - 1)}{r^n - 1}$.

3. Insert a geometric mean between 7 and 63.

We have $\dfrac{x}{7} = \dfrac{63}{x}$;

whence $x^2 = 7 \cdot 63 = 441$.

$\therefore x = \pm\sqrt{441} = \pm 21$.

Therefore the series is either 7, 21, 63, or 7, − 21, 63, r being 3 in the first case and − 3 in the second case.

221. Formulas of Geometric Progressions. Only the three fundamental formulas of §§ 218 and 219 need to be memorized. From them the following may be deduced, teachers using their judgment as to the cases to be considered.

No.	Given	Required	Result
1	$a\ r\ n$	l	$l = ar^{n-1}$
2	$a\ r\ s$		$l = \dfrac{a + (r-1)s}{r}$
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$
4	$r\ n\ s$		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}$
5	$a\ r\ n$	s	$s = \dfrac{a(r^n - 1)}{r - 1}$
6	$a\ r\ l$		$s = \dfrac{rl - a}{r - 1}$
7	$a\ n\ l$		$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
8	$r\ n\ l$		$s = \dfrac{lr^n - l}{r^n - r^{n-1}}$
9	$r\ n\ l$	a	$a = \dfrac{l}{r^{n-1}}$
10	$r\ n\ s$		$a = \dfrac{(r-1)s}{r^n - 1}$
11	$r\ l\ s$		$a = rl - (r-1)s$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\dfrac{l}{a}}$
14	$a\ n\ s$		$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0$
15	$a\ l\ s$		$r = \dfrac{s-a}{s-l}$
16	$n\ l\ s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0$

Exercise 102. Problems in Geometric Progressions

All Examples written

1. Find the fifth term of 3, 6, 12, ···; the tenth term.
2. Find the sixth term of 4, 12, 36, ···; the twelfth term.
3. Find the eighth term of 32, 16, 8, ···; the tenth term.
4. Find the ninth term of 4, -16, 64, ···; the tenth term.
5. Find the tenth term of 1, $\frac{1}{2}$, $\frac{1}{4}$, ···; of 1, 2, 4, ···.
6. Find the twelfth term of -3, 9, -27, ···; the thirteenth term; the twentieth term.

Find the sum of the following series:

7. 3, 6, 12, ··· to five terms; to ten terms.
8. 4, 12, 36, ··· to six terms; to twelve terms.
9. 2, -4, 8, ··· to five terms; to six terms.
10. 3, -9, 27, ··· to eight terms; to nine terms.
11. 1, $\frac{1}{2}$, $\frac{1}{4}$, ··· to eight terms; to fifteen terms.
12. Insert a geometric mean between 4 and 25 and find the sum of the three terms.
13. A geometric series is 3, $3r$, $3r^2$, 1029. Find r, and thus insert two geometric means between 3 and 1029. (Extract the root by factoring.)
14. Insert two geometric means between 2 and 1458.
15. Insert three geometric means between 2 and 32.
16. Given $a = 8$, $r = 2$, $s = 248$, find l and write the series.
17. Given $a = 32$, $r = \frac{1}{2}$, $n = 6$, find l and s.
18. Given $l = 3$, $r = \frac{1}{7}$, $n = 5$, find a and s.
19. Given $s = -42$, $a = -64$, $r = -\frac{1}{2}$, find l and write the series. How many terms are there?
20. Given $r = 6$, $n = 5$, $l = 1296$, find a and s.
21. The first term of a geometric series is 5 and the ratio is 2; what term is 1280?

222. Infinite Geometric Series. When r is a proper fraction the successive terms become numerically smaller and smaller. By taking n large enough we can therefore make the nth term as small as we please, bringing it nearer and nearer to zero.

Thus in the series $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$, the terms are getting smaller and smaller. If we take $n = 14$, we shall have $l = \frac{1}{2048}$, a small fraction; and if we take $n = 21$, we shall have $l = \frac{1}{2097152}$, a very small fraction.

Since $r < 1$, we may avoid negative terms by writing $s = \dfrac{ar^n - a}{r - 1}$ in the form $\dfrac{a - ar^n}{1 - r}$; and by taking n large enough we can make ar^n as near zero as we please, and can make s approach as near as we please to $\dfrac{a - 0}{1 - r}$, or $\dfrac{a}{1 - r}$. That is, $\dfrac{a}{1 - r}$ is the *limit* to which s approaches when r is a proper fraction.

For convenience it is usually said that $s = \dfrac{a}{1 - r}$, when r is a proper fraction and n is infinite.

The full statement is that the limit of s is $\dfrac{a}{1 - r}$, when r is a proper fraction and the number of terms increases without limit.

However far we extend the series, s lacks a little of being $\dfrac{a}{1 - r}$, but the further we extend it, the nearer s approaches this limit, the difference between s and $\dfrac{a}{1 - r}$ being less than any assigned positive quantity.

1. Find the sum of the infinite series $10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$.

Since $s = \dfrac{a}{1 - r}$, we have $s = \dfrac{10}{1 - \frac{1}{2}} = 20$. That is, the further we go in summing the series, the nearer the sum approaches 20.

2. Find the sum of the infinite series $9, -3, 1, -\frac{1}{3}, \cdots$.

Here $a = 9$, and $r = -\frac{1}{3}$.

Therefore $s = \dfrac{a}{1 - r} = \dfrac{9}{1 - (-\frac{1}{3})} = \dfrac{9}{\frac{4}{3}} = \dfrac{27}{4} = 6\frac{3}{4}$.

3. Find the sum of the infinite series $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \cdots$.

Here $a = \frac{1}{10}$, and $r = \frac{1}{10}$.

Therefore $s = \dfrac{a}{1 - r} = \dfrac{0.1}{1 - 0.1} = \dfrac{0.1}{0.9} = \dfrac{1}{9}$.

That is, the limit of the decimal fraction $0.111 \cdots$ is $\frac{1}{9}$.

4. Find the value of the fraction $0.72232323\cdots$.

Such a fraction is called a *recurring decimal* (or *repeating* or *circulating* decimal). It may be written

$$0.72 + \tfrac{23}{10000} + \tfrac{23}{1000000} + \cdots$$

in which 0.72 is not part of the infinite geometric series that follows.

In the series, $a = \frac{23}{10000}$ and $r = \frac{1}{100}$. Hence

$$s = \frac{a}{1-r} = \frac{\frac{23}{10000}}{1-\frac{1}{100}} = \frac{23}{9900}.$$

Add 0.72, the part of the decimal that does not recur, and

$$0.72 + \tfrac{23}{9900} = \tfrac{7151}{9900}.$$

If we reduce $\frac{7151}{9900}$ to a decimal fraction, we shall find that it equals $0.722323\cdots$, thus checking the work.

Exercise 103. Infinite Geometric Series

All Examples written

1. Find the sum of the infinite series $15, 5, 1\frac{2}{3}, \cdots$.

2. Find the sum of the infinite series $12, 3, \frac{3}{4}, \cdots$.

3. Find the sum of the infinite series $32, -16, 8, -4, \cdots$.

Find the value of the following recurring decimals:

4. $0.2727\cdots$.

5. $0.3030\cdots$.

6. $0.481481\cdots$.

7. $0.520520\cdots$.

8. $0.76565\cdots$.

9. $0.83421421\cdots$.

Find the sum of the following infinite series:

10. $100, 50, 25, \cdots$.

11. $99, 33, 11, \cdots$.

12. $160, 40, 10, \cdots$.

13. $625, 125, 25, \cdots$.

14. $111, 74, 49\frac{1}{3}, \cdots$.

15. $625, 250, 100, \cdots$.

16. $1250, 750, 450, \cdots$.

17. $4096, 3584, 3136, \cdots$.

18. If you take half of a line 4 in. long, and half of what is left, and so on, what is the limit of the sum of these parts?

19. If it were possible for a rubber ball to fall 10 ft. and bound back 5 ft., then to fall 5 ft. and bound back $2\frac{1}{2}$ ft., and to continue this forever, what is the limit of the distance through which the ball would pass?

CHAPTER XII

THE BINOMIAL THEOREM

223. Factorial. The product of the positive integers from one to any given number n, inclusive, is called *factorial n*.

That is, $1 \cdot 2 \cdot 3 \cdot 4 =$ factorial 4, and $1 \cdot 2 \cdot 3 \cdots n =$ factorial n.

There are two common symbols for factorial n, as follows: $\lfloor \underline{n}$ and $n\,!$, the former being more convenient to write and the latter more convenient to print. Thus $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = \lfloor \underline{5} = 5\,! = 120$.

224. Binomial Theorem. In order to develop the Binomial Theorem (§ 113) we may write the expansion of the first three powers of $a + b$ in the following form:

$$(a+b)^1 = a + b;$$

$$(a+b)^2 = a^2 + 2ab + \frac{2 \cdot 1}{2!} b^2;$$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3 \cdot 2}{2!} ab^2 + \frac{3 \cdot 2 \cdot 1}{3!} b^3.$$

We therefore infer that, in the case of $(a+b)^n$,

1. *The number of terms is greater by one than the exponent of the power to which the binomial is raised.*

2. *The exponent of a in the first term is n, and it decreases by one to the right.*

3. *The exponent of b in the first term is 0, and it increases by one to the right.*

4. *The coefficient of the first term is 1, and of the second term n.*

5. *The coefficient of each term after the first is found from the next preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.*

225. Proof of the Binomial Theorem. We know that the laws just stated hold for the third power, because we can obtain this power (§ 224) by actual multiplication.

Let us, for the moment, assume that they hold for the kth power, k being any positive integer. Then

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 + \cdots. \quad (1)$$

If we multiply both members of (1) by $a+b$ in the usual manner, we have

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{2!}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{3!}a^{k-2}b^3 + \cdots. \quad (2)$$

But (2) is exactly what we obtain if we expand $(a+b)^{k+1}$ by the Binomial Theorem.

Therefore, *if* the law holds for the kth power (that is, if (1) is true), it holds for the $(k+1)$th power, for we have shown this by actually multiplying (1) by $a+b$.

But the law *does* hold true for the *third* power (§ 224), and therefore it must hold true for the $(3+1)$th, or *fourth* power. Since it holds true for the fourth power, it must hold true for the $(4+1)$th, or fifth power, and so on for any positive integral power. Therefore, for the nth power,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots.$$

This is a method of proof known as *mathematical induction*.

Evidently we may interchange a and b. This will give us the last few terms of the series, just as we have now the first few, thus:

$$\cdots + \frac{n(n-1)}{2!}a^2b^{n-2} + nab^{n-1} + b^n.$$

This is illustrated in such a familiar case as $a^3 + 3a^2b + 3ab^2 + b^3$.

1. Expand $(a-b)^n$.

Evidently the even powers of $-b$ are positive and the odd powers negative. We therefore have

$$(a-b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-b)^n$.

2. Expand $(1+x)^n$.

Substituting 1 for a, and x for b, in $(a+b)^n$, we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)}{2!}x^{n-2} + nx^{n-1} + x^n.$$

3. Expand $(1-x)^n$.

In Ex. 2 put $-x$ for x, and we have

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

We cannot tell the sign of the last term unless we know whether n is odd or even. It is, however, $(-x)^n$.

4. Expand $(1+2a)^6$.

Substituting 1 for a, and $2a$ for b, in $(a+b)^n$, we have

$$(1+2a)^6 = 1 + 6(2a) + \frac{6\cdot5}{2!}(2a)^2 + \frac{6\cdot5\cdot4}{3!}(2a)^3 + \frac{6\cdot5\cdot4\cdot3}{4!}(2a)^4 + \frac{6\cdot5\cdot4\cdot3\cdot2}{5!}(2a)^5 + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6!}(2a)^6$$
$$= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6.$$

5. Expand to four terms $\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7$.

Substituting $\frac{2}{x}$ for a, and $-\frac{3x^2}{4}$ for b, in $(a+b)^n$, we have

$$\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7 = \left(\frac{2}{x}\right)^7 - 7\left(\frac{2}{x}\right)^6\left(\frac{3x^2}{4}\right) + 21\left(\frac{2}{x}\right)^5\left(\frac{3x^2}{4}\right)^2 - 35\left(\frac{2}{x}\right)^4\left(\frac{3x^2}{4}\right)^3 + \cdots$$
$$= \frac{128}{x^7} - \frac{336}{x^4} + \frac{378}{x} - \frac{945x^2}{4} + \cdots.$$

We could also have written this $(2x^{-1} - \frac{3}{4}x^2)^7$ and expanded, obtaining $128x^{-7} - 336x^{-4} + 378x^{-1} - 236\frac{1}{4}x^2 + \cdots$.

226. Formula for the rth term. We see by § 224 that the third term is $\dfrac{n(n-1)}{2!}a^{n-2}b^2$,

the fourth term is $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3, \cdots$,

and hence that the rth term is

$$\frac{n(n-1)(n-2)\cdots \text{to } (r-1) \text{ factors}}{(r-1)!}a^{n-(r-1)}b^{r-1}.$$

Find the eighth term of $\left(4-\dfrac{x^2}{2}\right)^{10}$.

Here $a = 4$, $b = -\frac{1}{2}x^2$, $n = 10$, and $r = 8$.

The eighth term is

$$\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{7!}(4)^{10-7}(-\tfrac{1}{2}x^2)^7, \text{ or } -60x^{14}.$$

Exercise 104. Binomial Theorem

Examples 1 to 3, oral — Examples 4 to 20, written

1. Expand $(1+x)^3$; $(1-x)^3$; $(a+2)^3$; $(2-a)^3$.
2. Expand $(a+b)^4$; $(a-b)^4$; $(a+1)^4$; $(1-a)^4$.
3. Expand $(a+b)^5$; $(a+1)^5$; $(a-b)^5$; $(1-a)^5$.

Expand the following expressions:

4. $(a+3b)^3$.
5. $(a-3b)^4$.
6. $(\frac{1}{2}a+\frac{2}{3}b)^4$.
7. $\left(\dfrac{3}{x}-\dfrac{x}{3}\right)^5$.
8. $(a^2+2b)^6$.
9. $(2a^2-3b)^6$.
10. $(\frac{1}{2}a-\frac{2}{3}b)^6$.
11. $\left(\dfrac{2}{x}-\dfrac{x^2}{5}\right)^6$.
12. $(a^{-2}+1^{-4})^3$.
13. $(2a^{-3}-x^{-5})^6$.
14. $(\frac{1}{2}\sqrt{x}-0.1\sqrt{y})^7$.
15. $\left(\sqrt[3]{\dfrac{x^2}{3}}-\sqrt{\dfrac{2}{x}}\right)^9$.
16. In $(3x-4y)^7$ find the fifth term; the sixth.
17. In $(2x+3)^8$ find the ninth term; the tenth.
18. In $(3x-\frac{1}{2}y^2)^{15}$ find the tenth term; the twelfth.
19. In $(5a^2-3b^2)^{14}$ find the middle term.
20. In $(a^{\frac{1}{2}}+b^{\frac{1}{3}})^{12}$ find the middle term.

227. Convergent Series. A series in which the sum of the terms, as the number of terms is indefinitely increased, approaches some fixed finite value as a limit, is called a *convergent series.*

For example, as shown in § 222, when the number of terms is indefinitely increased, the limit approached by $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is $\frac{1}{1 - \frac{1}{2}}$, or 2. This is therefore a convergent series.

But the series 2, 4, 6, $\cdots$ is not a convergent series, because, when the number of terms is indefinitely increased, $2 + 4 + 6 + \cdots$ becomes infinite. Such a series is called a *divergent series.*

228. Binomial Theorem, any Exponent. The Binomial Theorem is true for any exponent, integral or fractional, positive or negative, provided the series that results is convergent.

The proof of this fact is not suited to elementary algebra, and it has to be assumed at this time.

1. Expand to four terms $(x + y)^{\frac{1}{2}}$.

Substituting in the expansion of $(a + b)^n$, we have

$$(x + y)^{\frac{1}{2}} = x^{\frac{1}{2}} + \tfrac{1}{2}x^{\frac{1}{2}-1}y + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^{\frac{1}{2}-2}y^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^{\frac{1}{2}-3}y^3 + \cdots$$

$$= x^{\frac{1}{2}} + \frac{y}{2x^{\frac{1}{2}}} - \frac{y^2}{8x^{\frac{3}{2}}} + \frac{y^3}{16x^{\frac{5}{2}}} - \cdots.$$

2. Expand to three terms $\sqrt[3]{9}$.

$$\sqrt[3]{9} = \sqrt[3]{8+1} = (8+1)^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}} + \tfrac{1}{3}\cdot 8^{\frac{1}{3}-1} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}8^{\frac{1}{3}-2} + \cdots$$

$$= 2 + \tfrac{1}{12} - \tfrac{1}{288} + \cdots = 2.079+.$$

3. Expand to three terms $(2x - \frac{1}{8}y)^{-\frac{2}{3}}$.

Substituting in the expansion of $(a + b)^n$, we have

$$(2x - \tfrac{1}{8}y)^{-\frac{2}{3}} = (2x)^{-\frac{2}{3}} + (-\tfrac{2}{3})(2x)^{-\frac{2}{3}-1}(-\tfrac{1}{8}y) + \frac{(-\frac{2}{3})(-\frac{2}{3}-1)}{2!}(2x)^{-\frac{2}{3}-2}(-\tfrac{1}{8}y)^2 + \cdots$$

$$= (2x)^{-\frac{2}{3}} + \tfrac{2}{6}(2x)^{-\frac{5}{3}}y + \tfrac{5}{81}(2x)^{-\frac{8}{3}}y^2 + \cdots.$$

4. Expand to four terms $(1-x)^{-1}$.

$$(1-x)^{-1}=1+(-1)\cdot 1^{-1-1}(-x)+\frac{(-1)(-1-1)}{2!}1^{-1-2}(-x)^2+\cdots$$
$$=1+x+x^2+x^3+\cdots.$$

As a special case, suppose $x=2$. We then have

$$(1-2)^{-1}=1+2+2^2+2^3+\cdots.$$

But
$$(1-2)^{-1}=\frac{1}{1-2}=\frac{1}{-1}=-1.$$

Therefore it would seem that $-1=1+2+2^2+2^3+\cdots$, which is absurd. The trouble is easily seen, however, for the series $1+2+2^2+2^3+\cdots$ is manifestly not convergent, and the Binomial Theorem does not hold for such cases.

Exercise 105. Binomial Theorem, any Exponent

All Examples written

1. Expand to four terms: $(1+n)^{\frac{1}{2}}$; $(1+n)^{-2}$.

2. Expand $(1+x)^{-\frac{1}{2}}$ to four terms, and write the result with fractions and radical signs instead of negative and fractional exponents.

Expand the following expressions to four terms:

3. $(x-1)^{-3}$.
4. $(a-x)^{-4}$.
5. $(2a+1)^{-3}$.
6. $(1+x)^{\frac{3}{2}}$.
7. $(1-x)^{-\frac{3}{2}}$.
8. $(a-2x)^{-\frac{3}{2}}$.
9. $(a^2b^{-2}+1)^{\frac{1}{2}}$.
10. $(8a^{-3}-1)^{\frac{1}{2}}$.
11. $(8x^{-3}-27)^{-\frac{2}{3}}$.
12. In $(a-b)^{-5}$ find the fourth term.
13. In $(a-3)^{-6}$ find the fifth term; the sixth.
14. In $(a+x)^{\frac{1}{2}}$ find the third term; the fourth.
15. In $(2a-1)^{\frac{1}{2}}$ find the third term; the fifth.
16. Expand to three terms: $\sqrt{5}$, $\sqrt[3]{28}$, $\sqrt[5]{33}$.
17. In $\left(x-\frac{1}{x}\right)^{12}$, what is the term that does not contain x?
18. Find the square root of 10 to three decimal places by the Binomial Theorem. Verify the result by finding the square root by the ordinary method.

CHAPTER XIII

LOGARITHMS

229. Logarithm. The power to which a given number, called the *base*, must be raised to equal another number is called the *logarithm* of this other number.

For example, since $10^3 = 1000$,

therefore, to the base 10, 3 = the logarithm of 1000.

In general, if $b^x = N$,

then, to the base b, x = the logarithm of N.

230. Symbolism. For "logarithm of N" it is customary to write $\log N$. If we wish to specify $\log N$ to the base b, we write $\log_b N$, reading this, "logarithm of N to the base b."

It is evident from § 229 that, to the base 10, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, $\log 10{,}000$ equals 4, and so on.

231. Base. We may take various bases for systems of logarithms, but for practical calculation 10 is taken.

Thus, since $2^3 = 8$, therefore $\log_2 8 = 3$.

If we take 10 as the base, since $10^2 = 100$, and $10^3 = 1000$, we know that the logarithm of any number between 100 and 1000 must lie between 2 and 3. Hence, for example, we know that $\log 475$ is $2 +$ some fraction.

By using 10 as the base, therefore, we can tell at once the integral part of the logarithm of any number.

When we write $\log 475$ we mean $\log_{10} 475$, the base 10 being understood unless another base is specified.

Since $\log 10 = 1$, because $10^1 = 10$,

and $\log 1 = 0$, because $10^0 = 1$,

and $\log \frac{1}{10} = -1$, because $10^{-1} = \frac{1}{10}$,

we see that *the logarithm of the base is 1, the logarithm of 1 is zero, and the logarithm of a proper fraction is negative.*

Exercise 106. Logarithms

1. Since $2^4 = 16$, what is $\log_2 16$?
2. Since $4^2 = 16$, what is $\log_4 16$?
3. Since $10^5 = 100{,}000$, what is $\log 100{,}000$?

Write the following logarithms:

4. $\log_2 32$.
5. $\log_2 128$.
6. $\log_2 256$.
7. $\log_2 512$.
8. $\log_3 9$.
9. $\log_3 81$.
10. $\log_4 64$.
11. $\log_5 25$.
12. $\log_6 216$.
13. $\log_7 49$.
14. $\log_8 512$.
15. $\log_9 729$.
16. $\log 1000$.
17. $\log 10{,}000$.
18. $\log 1{,}000{,}000$.
19. $\log 10{,}000{,}000$.
20. Since $10^{-1} = \frac{1}{10}$, or 0.1, what is $\log 0.1$?
21. What is $\log \frac{1}{100}$, or $\log 0.01$? $\log 0.001$? $\log 0.0001$?
22. Between what integers is $\log 927$? $\log 1756$? $\log 42{,}345$?
23. Between what negative integers is $\log 0.02$? $\log 0.007$?
24. To the base 2, write the logarithms of 2, 4, 64, $\frac{1}{2}$, $\frac{1}{8}$.
25. To the base 3, write the logarithms of 27, 243, $\frac{1}{3}$, $\frac{1}{27}$.
26. To the base 10, write the logarithms of $\frac{1}{10000}$, $\frac{1}{1000000}$, 0.00001, and 100,000,000.

Write the integers between which the logarithms of the following numbers lie:

27. 83.
28. 83.5.
29. 127.
30. 127.96.
31. 4237.
32. 4237.8.
33. 42,756.
34. 42,756.95.

Show that the following statements are true:

35. $\log_2 8 + \log_2 16 + 3\log_2 32 - 2\log_2 64 = 10$.
36. $\log_3 9 + 4\log_3 81 + 2\log_3 3 + \log_3 \frac{1}{9} + \log_3 \frac{1}{27} = 15$.
37. $\log_4 4 + 7\log_4 16 + 10\log_4 64 + \log_4 \frac{1}{4} + \log_4 \frac{1}{16} = 42$.
38. $\log 1 + \log 10 + \log 100 + \log 1000 + \log 0.001 = 3$.
39. $7\log 1 + 9\log 10{,}000 + \log 0.1 + \log 0.000001 = 29$.
40. $\log 10{,}000 - \log 1000 + \log 100{,}000 - \log 100 = 4$.

232. Logarithm of a Product. *The logarithm of the product of two factors equals the sum of the logarithms of the factors.*

Let A and B be the factors, and x and y their logarithms. Then, taking 10 as the base, and remembering that $x = \log A$ and $y = \log B$, we have

$$A = 10^x,$$

and

$$B = 10^y.$$

Therefore

$$AB = 10^{x+y},$$

and therefore

$$\log AB = x + y$$
$$= \log A + \log B.$$

The proof is the same if any other base is taken.

For if $A = b^x$, and $B = b^y$, we see that $x = \log_b A$, and $y = \log_b B$. Furthermore, $AB = b^{x+y}$, whence $\log_b AB = x + y = \log_b A + \log_b B$. This proof is the same as the one given above, with 10 replaced by b.

The proposition is true for the product of any number of factors, the proof being evidently the same. Thus,

$$\log ABC = \log A + \log B + \log C,$$
$$\log ABCD = \log A + \log B + \log C + \log D,$$

and so on for any number of factors.

233. Logarithm of a Quotient. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

For if $x = \log A$, then

$$A = 10^x,$$

and if $y = \log B$, then

$$B = 10^y.$$

Therefore

$$\frac{A}{B} = 10^{x-y},$$

and therefore

$$\log \frac{A}{B} = x - y$$
$$= \log A - \log B.$$

It is therefore seen from §§ 232 and 233 that if we know the logarithms of all numbers, we can find the logarithm of a product by addition and the logarithm of a quotient by subtraction. If we can then find the numbers of which these results are the logarithms, we shall have solved our problems in multiplication and division by merely adding and subtracting.

234. Logarithm of a Power. *The logarithm of a power of a number equals the logarithm of the number multiplied by the exponent.*

If $x = \log A$, then $A = 10^x$.

Raising to the pth power, $A^p = 10^{px}$.

Hence $\log A^p = px$

$= p \log A$.

This is easily seen to be true for any base whatever, either by letting $A = b^x$ and proceeding as above, or by taking special numbers. Thus if we take the base 2, instead of the base 10 as above, we have,

since $2^5 = 32$,

$\log_2 32 = 5$.

And since $(2^5)^2 = 32^2 = 1024$,

we have $\log 1024 = 2 \times 5$

$= 2 \log_2 32$.

That is, $\log_2 32^2 = 2 \log_2 32$.

235. Logarithm of a Root. *The logarithm of a root of a number equals the logarithm of the number divided by the index of the root.*

If $x = \log A$, then $A = 10^x$.

Taking the rth root, $A^{\frac{1}{r}} = 10^{\frac{x}{r}}$.

Hence $\log A^{\frac{1}{r}} = \frac{x}{r}$

$= \frac{\log A}{r}$.

As in § 234, this can easily be proved for any base.

From §§ 234 and 235 we see that the raising of a number to any power, integral or fractional, reduces itself to multiplying the logarithm by the exponent, integral or fractional, and then finding the number of which the result is the logarithm.

Therefore the operations of multiplication, division, raising to powers, and extracting roots will be greatly simplified, as already stated in § 233, if we can find the logarithms of numbers, and this will next be considered.

236. Characteristic and Mantissa. Usually a logarithm consists of an integer plus a decimal fraction. The integral part of a logarithm is called the *characteristic.*

The decimal part of a logarithm is called the *mantissa.*

The word "mantissa" means an addition, something attached, and hence has come to signify the decimal part that is attached to the integral part of a logarithm.

Thus if $\log 2353 = 3.37162$, the characteristic is 3 and the mantissa 0.37162. This means that $10^{3.37162} = 2353$, or that the 100,000th root of the 337,162d power of 10 is 2353, approximately.

237. Finding the Characteristic. Since we know that

$$10^3 = 1000 \quad \text{and} \quad 10^4 = 10{,}000,$$

therefore $$3 = \log 1000 \quad \text{and} \quad 4 = \log 10{,}000.$$

Hence the logarithm of a number between 1000 and 10,000 lies between 3 and 4, and is therefore 3 plus some fraction.

The same reasoning shows us that the logarithm of a number between 10,000 and 100,000 lies between 4 and 5.

Likewise, since

$$10^{-2} = 0.01 \quad \text{and} \quad 10^{-3} = 0.001,$$

therefore $$-2 = \log 0.01 \quad \text{and} \quad -3 = \log 0.001.$$

Hence the logarithm of a number between 0.001 and 0.01 lies between -3 and -2, and is therefore -3 plus some fraction. Hence the characteristic is -3.

Of course, instead of saying that log 1475 is 3 plus a fraction we might say that it is 4 minus a fraction; and instead of saying that log 0.007 is -3 plus a fraction we might say that it is -2 minus a fraction.

For convenience, *the mantissa is always taken as positive,* although the characteristic may be either positive or negative.

The reason for this custom is apparent. We shall find (§ 241) that there are tables of mantissas of logarithms conveniently arranged for use. If we were compelled to use negative mantissas, the tables would be much more cumbersome and the work more complicated. Since we see that the characteristic is easily found without the aid of tables, the fact that it may be either positive or negative does not present any serious difficulties.

238. Rule for the Characteristic. From the reasoning set forth in § 237 we deduce the following rule:

1. *The characteristic of a number greater than 1 is positive, and is one less than the number of integral places in the number.*

For example, $\log 75 = 1 +$ some mantissa,

$\log 472.8 = 2 +$ some mantissa,

and $\log 14{,}800.75 = 4 +$ some mantissa.

2. *The characteristic of a number between 0 and 1 is negative, and is one more than the number of zeros between the decimal point and the first significant figure in the decimal fraction.*

For example, $\log 0.02 = -2 +$ some mantissa,

and $\log 0.00076 = -4 +$ some mantissa.

The logarithm of a negative number is an imaginary number, and hence such logarithms are not used in computation.

239. The Negative Characteristic. If $\log 0.02 = -2 + 0.30103$, we cannot write it -2.30103, because this would mean that both mantissa and characteristic are negative. It is therefore written $\bar{2}.30103$, which means that only the characteristic 2 is negative.

That is, $\bar{2}.30103 = -2 + 0.30103$. We may also write it $0.30103 - 2$, or $8.30103 - 10$, or in any similar manner that will show that the characteristic is negative.

Exercise 107. Characteristics

Write the characteristics of the logarithms of:

1. 24.	9. 7235.	17. 0.8.	25. 0.0003.
2. 24.8.	10. 723.5.	18. 0.08.	26. 0.0033.
3. 248.	11. 72.35.	19. 0.88.	27. 0.0333.
4. 2.48.	12. 7.235.	20. 0.885.	28. 0.3333.
5. 2480.	13. 72,350.	21. 0.005.	29. 0.0303.
6. 2485.	14. 0.7235.	22. 0.0051.	30. 1.0303.
7. 2485.7.	15. 0.07235.	23. 0.0005.	31. 2.0303.
8. 2485.72.	16. 0.007235.	24. 0.00051.	32. 10.0303.

240. Mantissa independent of Decimal Point. It can be shown that $10^{3.37107} = 2350$, whence $\log 2350 = 3.37107$.

Dividing by 10 we have

$$10^{3.37107-1} = 235, \quad \text{whence} \quad \log 235 = 2.37107.$$

Dividing the members of the first equation by 10^4, or 10,000, we have

$$10^{3.37107-4} = 0.235, \quad \text{whence} \quad \log 0.235 = \bar{1}.37107.$$

That is, the mantissas are the same for log 2350, log 235, log 0.235, and so on, wherever the decimal point is placed.

The mantissa of the logarithm of a number is unchanged by any change in the position of the decimal point of the number.

This is a fact of great importance, for if the table of logarithms, which we shall soon describe, gives us the mantissa of log 235, we know that we may use the same mantissa for log 0.00235, log 2.35, and so on.

Exercise 108. Mantissas and Characteristics

Given log 625 = 2.7959, find:

1. log 62.5.
2. log 6.25.
3. log 0.625.
4. log 6250.
5. log 62,500.
6. log 0.0625.
7. log 625,000.
8. log 0.00625.
9. log 6,250,000.

Given log 16,630 = 4.2209, find:

10. log 1.663.
11. log 16.63.
12. log 166.3.
13. log 0.1663.
14. log 0.01663.
15. log 0.001663.
16. log 166,300.
17. log 1,663,000.
18. log 16,630,000.

Given log 9.154 = 0.9616, find:

19. log 91.54.
20. log 915.4.
21. log 0.9154.
22. log 0.09154.
23. log 9154.
24. log 915,400.

Given log π = 0.4971, find:

25. log π^2.
26. log π^8.
27. log π^4.
28. log π^{10}.
29. log $\pi^{\frac{1}{2}}$.
30. log $\pi^{\frac{1}{3}}$.
31. log $\sqrt[4]{\pi}$.
32. log $\sqrt[5]{\pi}$.

241. Table of Logarithms. A table of logarithms to four decimal places is given on pages 248 and 249. It gives the mantissas for all numbers less than 1000, the decimal points being omitted.

Such a table is called a "four-place table."

Tables in which the mantissas are given to more than four decimal places are used when greater accuracy is required, but for ordinary computations a four-place table is usually sufficient to give results that are accurate to four significant figures.

In the table the numbers are given under N and the tenths under the columns headed 0, 1, 2, $\cdots$ 9.

Since only the mantissas are given, *always write the characteristic before looking up the mantissa, so that it shall not be forgotten.*

242. Finding the Logarithm of a Number. The following examples explain the use of the table:

1. Find the logarithm of 73.4.

First write the characteristic, 1.

In column N look for the first two figures, 73.

Then look to the right of 73 and in column 4. Here the mantissa is found to be 0.8657.

Hence $\log 73.4 = 1.8657$.

2. Find the logarithm of 4236.

First write the characteristic, 3.

Then notice that 4236 is between 4230 and 4240, and is 0.6 of the way from 4230 to 4240.

Hence we may assume that $\log 4236$ is approximately 0.6 of the way from $\log 4230$ to $\log 4240$.

In column N look for the first two figures, 42.

Then look to the right of 42 and in columns 3 and 4. Here we find that

$$\log 4240 = 3.6274$$
$$\log 4230 = \underline{3.6263}$$
$$0.6 \text{ of } 0.0011 = 0.0007, \text{ nearly.}$$

Adding 0.0007 to 3.6263, we have $\log 4236 = 3.6270$.

This plan of finding the logarithm of a number greater than those given in the tables is called *interpolation.*

3. Find the logarithm of 0.0002705.

First write the characteristic, -4.

Proceeding as in Ex. 2, we have

$$\log 0.000271 = 0.4330 - 4$$
$$\log 0.000270 = 0.4314 - 4$$
$$0.5 \text{ of } \overline{0.0016} = 0.0008$$

Adding 0.0008 to $0.4314 - 4$, we have $0.4322 - 4$.

We may write $\log 0.0002705$ with the -4 at the left, thus: $\bar{4}.4322$. When we have subtractions to perform, however, it is less confusing to place the negative characteristic at the right as shown above. It is also convenient to write the negative characteristic at the right in performing other operations on logarithms.

4. Find the logarithm of 7.

Since the mantissa of $\log 7$ is the same as that of $\log 700$, we look for 70 in column N and under column 0. Hence $\log 7 = 0.8451$.

Exercise 109. Finding Logarithms

Using the table on pages 248 and 249, find the logarithms of:

1. 24.	**16.** 4.	**31.** 22.	**46.** 182.
2. 32.	**17.** 7.	**32.** 222.	**47.** 182.3.
3. 76.	**18.** 0.4.	**33.** 222.2.	**48.** 182.9.
4. 48.	**19.** 0.7.	**34.** 2222.	**49.** 18.29.
5. 60.	**20.** 0.44.	**35.** 0.22.	**50.** 1.829.
6. 100.	**21.** 0.77.	**36.** 0.022.	**51.** 427.
7. 200.	**22.** 1.44.	**37.** 0.222.	**52.** 4275.
8. 270.	**23.** 1.77.	**38.** 0.277.	**53.** 42.75.
9. 275.	**24.** 17.7.	**39.** 3.270.	**54.** 4.275.
10. 2756.	**25.** 177.	**40.** 0.5000.	**55.** 427.5.
11. 27.56.	**26.** 1770.	**41.** 0.5500.	**56.** 42,750.
12. 275.60.	**27.** 1775.	**42.** 0.5550.	**57.** 53,750.
13. 27,560.	**28.** 17,750.	**43.** 0.5555.	**58.** 50,000.
14. 32,450.	**29.** 25,300.	**44.** 55,500.	**59.** 50,050.
15. 41,270.	**30.** 25,350.	**45.** 55,550.	**60.** 75,080.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

243. Antilogarithm. The number corresponding to a given logarithm is called an *antilogarithm.*

Thus if log 676 is 2.8299, the antilogarithm of 2.8299 is 676.

244. Finding Antilogarithms. Antilogarithms are found from the table by looking for the number corresponding to the given mantissa, and locating the decimal point according to the characteristic.

1. Find the antilogarithm of 3.4265.

Looking for the mantissa 0.4265, we find that it is opposite 26 in column N and under column 7. It is therefore the mantissa of 267.

Since the characteristic is 3, there must be four integral places in the antilogarithm. Hence the antilogarithm must be 2670.

Therefore the antilogarithm of 3.4265 is 2670.

2. Find the antilogarithm of $\bar{2}.8404$.

Looking for the mantissa 0.8404, we do not find it in the table. We find 0.8401 and 0.8407, their difference being 0.0006. The difference between 0.8401 and 0.8404 is 0.0003. Hence 0.8404 is $\frac{3}{6}$ of the way from 0.8401 to 0.8407.

Hence the antilogarithm of 0.8404 must be approximately $\frac{3}{6}$ of the way from the antilogarithm of 0.8401 to that of 0.8407.

We therefore write

$$\text{antilog } \bar{2}.8407 = 0.06930$$
$$\text{antilog } \bar{2}.8401 = \underline{0.06920}$$
$$\tfrac{3}{6} \text{ (or } \tfrac{1}{2}\text{) of } \overline{0.00010} = 0.00005.$$

Adding 0.00005 to 0.06920 we have

$$\text{antilog } \bar{2}.8404 = 0.06925.$$

3. Find the antilogarithm of 0.3664.

Looking for the mantissa 0.3664, we find that it lies between 0.3655 and 0.3674, whose difference is 0.0019. Since $0.3664 - 0.3655 = 0.0009$, the given mantissa is $\frac{9}{19}$ of the way from 0.3655 to 0.3664. But antilog $0.3655 = 2.32$, and antilog 0.3674 is 2.33. Adding $\frac{9}{19}$ of the difference to 2.32, we have 2.325, the antilogarithm required.

4. Find the antilogarithm of 7.9050.

The mantissa is evidently $\frac{3}{6}$ of the way from the mantissa for 803 and that for 804. Hence it is the mantissa for 8035. The characteristic being 7, the antilogarithm is 80,350,000.

Exercise 110. Antilogarithms

Using the table, find the antilogarithms of:

1. 0.4771.	**11.** 0.1945.	**21.** 0.0000.	**31.** 0.7782 − 1.
2. 1.5682.	**12.** 1.2266.	**22.** 5.0000.	**32.** 0.7864 − 2.
3. 3.8451.	**13.** 2.8212.	**23.** 2.7408.	**33.** 0.7668 − 3.
4. $\bar{1}$.8865.	**14.** $\bar{2}$.8296.	**24.** 3.7406.	**34.** 0.8028 − 4.
5. 0.5065.	**15.** $\bar{4}$.8398.	**25.** $\bar{2}$.7410.	**35.** 9.8096 − 10.
6. 2.5211.	**16.** 4.8397.	**26.** 3.7735.	**36.** 9.8235 − 10.
7. 4.5977.	**17.** 1.8845.	**27.** 2.2620.	**37.** 8.8306 − 10.
8. 3.8785.	**18.** 2.8844.	**28.** 3.4210.	**38.** 8.8500 − 10.
9. $\bar{2}$.9380.	**19.** $\bar{2}$.8846.	**29.** $\bar{1}$.7280.	**39.** 8.8503 − 10.
10. $\bar{3}$.9741.	**20.** 3.8851.	**30.** $\bar{2}$.6666.	**40.** 7.9996 − 10.

41. If the logarithm of the product of two numbers is 3.5211, what is the product of the numbers?

42. If the logarithm of the quotient of two numbers is 2.8370, what is the quotient of the numbers?

43. If the logarithm of the square of a certain number is 2.1584, what is the square of the number? What is the number?

44. If the logarithm of the square root of a certain number is 1.3979, what is the square root of the number? What is the number?

45. If we wish to multiply 277 by 49.8, what logarithms do we need? Find these logarithms from the table.

46. If we know that the logarithm of a certain result that we are seeking is 3.8293, what is the result?

47. There is a certain number such that the logarithm of its square is 3.8062. What is the number?

48. If the logarithm of the cube root of a certain number is 0.6551, what is the logarithm of the number? What is the cube root of the number?

245. Computation by Logarithms. Since we have learned (§§ 232–235) how to find the logarithms of products, quotients, powers, and roots, and (§ 244) how to find antilogarithms, we may now consider a few typical problems in computation.

1. Find the product of 247 and 3.95.

$$\begin{aligned} \log 247 &= 2.3927 \\ \log 3.95 &= \underline{0.5966} \\ & \quad 2.9893 = \log 975.8. \end{aligned}$$

If we multiply 247 by 3.95 the product is found to be 975.65, or 975.7, if we carry the result to only four significant figures. Our last figure is therefore 1 too great. This shows us that computations by logarithms give, in general, results that are only approximately correct. The approximation is closer when we use larger tables. In all work in this book results should be carried to four significant figures only, and it should be understood that the last figure may be incorrect.

2. Find the product of 37.2, 0.416, and $-$ 3.275.

Since we cannot find the logarithm of a negative number, we proceed to find the product of 37.2, 0.416, and 3.275, prefixing the negative sign to the result. This we have the right to do because the product of two positive quantities is positive, and when this product is multiplied by a negative number, the result is negative.

$$\begin{aligned} \log 37.2 &= 1.5705 \\ \log 0.416 &= 0.6191 - 1 \\ \log 3.275 &= \underline{0.5152} \\ & \quad 2.7048 - 1 \\ & = 1.7048 = \log 50.68. \end{aligned}$$

Hence the product is $-$ 50.68.

If we multiply in the ordinary way, the result is found to be $-$ 50.68128. The result obtained by logarithms is therefore correct to four significant figures.

3. Find the quotient of $17.28 \div 1.44$.

$$\begin{aligned} \log 17.28 &= 1.2375 \\ \log 1.44 &= \underline{0.1584} \\ & \quad 1.0791 = \log 12, \text{ nearly.} \end{aligned}$$

Hence $17.28 \div 1.44 = 12$.

In this case we see that $\log 12 = 1.0792$, while the logarithm that is found is 1.0791, an error of 1 appearing in the fourth decimal place.

4. Find the quotient of $62.5 \div 0.025$.

$$\begin{aligned} \log 62.5 &= 1.7959 \\ \log 0.025 &= \underline{0.3979 - 2} \\ & \quad 1.3980 + 2 \\ &= 3.3980 = \log 2500, \text{ nearly.} \end{aligned}$$

Hence $62.5 \div 0.025 = 2500$.

Here again the fourth decimal place is incorrect. If we had been using a six-place table, this error would not have appeared. We should then have found that

$$\begin{aligned} \log 62.5 &= 1.795880 \\ \log 0.025 &= \underline{0.397940 - 2} \\ & \quad 1.397940 + 2 \\ &= 3.397940 = \log 2500, \text{ exactly.} \end{aligned}$$

For ordinary computations, however, in which we need only the first three or four figures of a result, the table on pages 248 and 249 is sufficient.

5. Find the value of 0.0048^3.

$$\begin{aligned} \log 0.0048 &= 0.6812 - 3 \\ & \quad \underline{\quad\quad 3} \\ & \quad 2.0436 - 9 \\ &= 0.0436 - 7 = \log 0.0000001106. \end{aligned}$$

The result by actual multiplication is 0.000000110592.

6. Find the value of $\sqrt[7]{2}$.

$$\begin{aligned} \log 2 &= 0.3010. \\ \tfrac{1}{7} \log 2 &= 0.0430 = \log 1.106. \end{aligned}$$

Hence $\sqrt[7]{2} = 1.104$, to four decimal places.

7. Find the value of $\sqrt[3]{2.4 \times 3.8 \times 0.0347}$.

$$\begin{aligned} \log 2.4 &= 0.3802 \\ \log 3.8 &= 0.5798 \\ \log 0.0347 &= \underline{0.5403 - 2} \\ & \quad 1.5003 - 2 \\ &= 2.5003 - 3 \\ \tfrac{1}{3} \text{ of } (2.5003 - 3) &= 0.8334 - 1 \\ &= \log 0.6814. \end{aligned}$$

Hence $\sqrt[3]{2.4 \times 3.8 \times 0.0347} = 0.6814$.

It will be noticed that it is easier to place the negative characteristic at the right. When we come to divide by 3 we avoid fractions by writing $2.5003 - 3$ instead of $1.5003 - 2$.

Exercise 111. Computations by Logarithms

Perform the following computations by logarithms:

1. 3.67×28.4.
2. 2.57×426.
3. 40.7×90.2.
4. 309×208.
5. 27×4762.
6. 39×289.7.
7. 56×48.92.
8. 73×5.176.
9. 8×0.1728.
10. 9×0.0146.
11. $7.9 \div 6.7$.
12. $15.7 \div 8.3$.
13. $42.8 \div 0.71$.
14. $0.007 \div 0.83$.
15. $0.062 \div 0.09$.
16. $82.83 \div 0.7$.
17. $7.009 \div 9.9$.
18. $7 \div 3.142$.
19. $9 \div 31.47$.
20. $0.6 \div 3.14$.
21. $\sqrt{2}$.
22. $\sqrt[3]{2}$.
23. $\sqrt[4]{7}$.
24. $\sqrt[5]{8}$.
25. $\sqrt[6]{128}$.
26. $\sqrt[7]{147.6}$.
27. $\sqrt[8]{0.0007}$.
28. 42.37^{7}.
29. 5.107^{18}.
30. 0.0076^{20}.
31. Find the value of $\sqrt{2.74 \times 42.95 \times 617.8}$.
32. Find the value of $\sqrt[3]{0.7 \times 0.0763 \times 128.4}$.
33. Find the value of $4.76 \times 49.35 \times 72.86 \times 0.07$.

Perform the following multiplications:

34. 4.389×0.000728.
35. 29.76×0.000047.
36. 0.472×0.006234.
37. -29.8×47.63.
38. $-47.82 \times (-2.79)$.
39. $-2.678 \times (-0.0073)$.

Perform the following divisions:

40. $\dfrac{27.73}{42.81}$.
41. $\dfrac{0.6987}{3.427}$.
42. $\dfrac{276.9}{0.007342}$.
43. $\dfrac{0.08193}{47.99}$.
44. $\dfrac{0.6398}{0.4926}$.
45. $\dfrac{0.0006872}{0.5283}$.

Perform the following operations:

46. $42^{\frac{2}{3}}$.
47. $368^{\frac{1}{3}}$.
48. $14.92^{\frac{3}{4}}$.
49. $287.9^{\frac{5}{6}}$.
50. $3.142^{\frac{2}{5}}$.
51. $0.0072^{\frac{3}{5}}$.
52. $(-21)^{\frac{2}{3}}$.
53. $(-7.21)^{\frac{3}{5}}$.
54. $(-2.96)^{\frac{1}{7}}$.

246. Cologarithm. The logarithm of the reciprocal of a number is called the *cologarithm* of the number.

The cologarithm of x is expressed thus: $\operatorname{colog} x$.

Since $\operatorname{colog} x = \log \frac{1}{x} = \log 1 - \log x = 0 - \log x$, it is evident that $\operatorname{colog} x = -\log x$.

We may write this $10 - \log x - 10$, thus avoiding negative mantissas. Hence

$$\operatorname{colog} 2 = 10 - \log 2 - 10 = 10 - 0.3010 - 10 = \bar{9}.6990 - 10.$$

We may write this $0.6990 - 1$, $\bar{1}.6990$, or $9.6990 - 10$.

247. Use of the Cologarithm. Since to divide by a number is the same as to multiply by its reciprocal, *instead of subtracting the logarithm of a number we may add its cologarithm.*

The cologarithm is easily written down by looking at the logarithm in the table. Thus, since $\log 21 = 1.3222$, we find colog 21 by subtracting this from $10.0000 - 10$, or from $9.999(10) - 10$. That is, beginning at the left we subtract the number represented by each figure from 9, except the right-hand one, which we subtract from 10. In full form we have

$$\begin{aligned} & 10.0000 - 10 = 9.999(10) - 10 \\ \log 21 = & \ 1.3222 \quad = 1.322 \ \ 2 \\ \operatorname{colog} 21 = & \qquad\quad 8.677 \ \ 8 - 10 = \bar{2}.6778 \end{aligned}$$

Similarly, we may find $\operatorname{colog} 0.03952$ thus:

$$\begin{aligned} & 9.999(10) - 10 \\ \log 0.03952 = & \ 8.596 \ \ 8 - 10 \\ \operatorname{colog} 0.03952 = & \ 1.403 \ \ 2 \quad = 1.4032 \end{aligned}$$

Practically, of course, we would look at log 0.03952 and subtract mentally, writing down 1.4032 at once.

For example, we see that the cologarithms given below are nearly as easily written as the logarithms:

Logarithms,	3.6042,	5.9605,	$7.4316 - 10$.
Cologarithms,	$6.3958 - 10$,	$4.0395 - 10$,	2.5644.

If we have to subtract a single logarithm, this is simpler than to find the cologarithm and add; but in complicated operations it is easier to use the cologarithm, as is seen on page 256.

Logarithms in which the characteristic is greater than 10 are not common, but in case they occur we may subtract from $11.0000 - 11$, $12.0000 - 12$, and so on, instead of subtracting from $10.0000 - 10$.

248. Advantages of the Cologarithm. The advantages of the cologarithm may be seen in simplifying the following expression:

$$\frac{17.28 \times 6.25 \times 16.9}{1.44 \times 0.25 \times 1.3}.$$

This case has been selected because it is one in which we can easily verify the answer by cancellation. By logarithms we have

$$\begin{aligned}
\log 17.28 &= 1.2375 \\
\log 6.25 &= 0.7959 \\
\log 16.9 &= 1.2279 \\
\operatorname{colog} 1.44 &= 9.8416 - 10 \\
\operatorname{colog} 0.25 &= 0.6021 \\
\operatorname{colog} 1.3 &= \underline{9.8861 - 10} \\
&\quad 23.5911 - 20 \\
&= 3.5911 = \log 3900.
\end{aligned}$$

Exercise 112. Computations by Logarithms

Perform the following computations by logarithms:

1. $\dfrac{17.28 \times 1.44}{0.288 \times 8.64}$.

2. $\dfrac{57.5 \times 0.64}{1.25 \times 3.2}$.

3. $\dfrac{12.8 \times 1.341}{14.9 \times 0.64}$.

4. $\sqrt{\dfrac{3 \times 4.1 \times 7.2}{0.2 \times 5.7 \times 9.8}}$.

5. $\sqrt{\dfrac{23 \times 4.8 \times 5.7}{1.9 \times 3.7 \times 0.3}}$.

6. $\sqrt[3]{\dfrac{31 \times 4.8 \times 294.3}{64.8 \times 7.123 \times 4.9}}$.

7. $\sqrt[3]{\dfrac{24 \times 37 \times 428.1}{53.2 \times 41.05 \times 3.7}}$.

8. $\left(\dfrac{0.548 \times 1.98}{39.6 \times 2.74}\right)^{\frac{1}{4}}$.

9. $\left(\dfrac{11.76 \times 1.022}{1.46 \times 39.2}\right)^{\frac{1}{4}}$.

10. $\left(\dfrac{87.68 \times 29.43 \times 51.80}{72.8 \times 0.4 \times 26.43}\right)^{\frac{3}{4}}$.

11. $\sqrt[4]{\dfrac{7.2 \times 3.8 \times 1.46}{1.82 \times 7.46 \times 83.04}}$.

12. $\sqrt[7]{\dfrac{2 \times 4.1 \times 0.7234 \times 96}{1.4 \times 3.82 \times 0.41}}$.

13. $\sqrt[8]{\dfrac{3 \times 42 \times 276 \times 0.2315}{2.7 \times 21.8 \times 375}}$.

14. $\sqrt[10]{\dfrac{1438 \times 2763 \times 1297}{3415 \times 3906 \times 929.8}}$.

CHAPTER XIV

COMPLEX NUMBERS

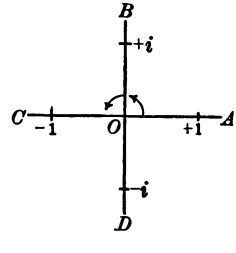
249. Imaginary Number. The square root of a negative number is called an *imaginary number.*

Because $\sqrt{-1}$ enters into all imaginary expressions of the form $\sqrt{-a}$, this being equal to $\sqrt{a}\sqrt{-1}$, it becomes necessary to study it with particular care. We have already, from pages 167–169, a fair knowledge of operations involving $\sqrt{-1}$, so we may now adopt a notation suggested on page 167, and use i for $\sqrt{-1}$. This will simplify our work, but we shall use $\sqrt{-1}$ when the meaning thereby becomes more plain.

250. Graphic Representation of an Imaginary Number. The term "imaginary" is an old one that was used before this kind of number was well understood. We shall see that $\sqrt{-1}$ is but little more imaginary than -1, but the name, being well established, will probably endure. To understand the nature of $\sqrt{-1}$, or i, it is advantageous to resort to graphic representation.

If we take two axes, AC and BD, intersecting at O, and lay off $+1$ on OA, as here shown, we see that if we multiply $+1$ by -1 the effect is to rotate $+1$ through 180° to the position -1.

That is, to multiply $+1$ by i^2, which is $(\sqrt{-1})^2$, or -1, is to rotate it through 180°. If to multiply $+1$ by i twice turns it through 180°, to multiply it by i once may properly be thought to rotate it through 90°, so that it will lie on OB. We therefore define i to be a unit measured from O on OB, upwards.



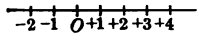
Similarly, we define $-i$ to be a unit measured downwards from O, on OD.

Hence i may be looked upon as a symbol of direction, just like $+$ and $-$. We then have $+3$, meaning 3 to the right, -3, meaning 3 to the left, $3i$ meaning 3 up, and $-3i$ meaning 3 down, all measured from O.

251. Graphic Representation of a Complex Number. A complex number being the sum of a real number and an imaginary, we may represent it graphically.

Before doing this we must consider what we mean by the sum of two real numbers.

If we add $+3$ and -2 the sum is $+1$. This is *numerically* smaller than either of the addends, but we know that the algebraic sum considers something besides mere length of lines; it considers also the direction of the lines.



That is, to add $+3$ and -2 we start at O and proceed 3 units to the right; then from the end of that line we proceed 2 units to the left; then *the sum is the distance from the initial point* (O) *to the terminal point.* In this case the absolute length of the resulting line is 1, and this is less than the absolute length of either line that is added.

Similarly, to add -2 and $+2$ we proceed from O to -2; then from -2 we go 2 units to the right. Again, *the sum is the distance from the initial point to the terminal point.*

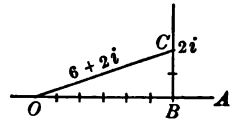
Similarly, to add $+2$ and $+2$ we proceed from O to $+2$; then from $+2$ we go 2 units to the right. The result is $+4$.

We therefore see that the idea of "sum" that we have stated holds for all these numbers.

If we have the complex number $6 + 2i$, we may represent it graphically as follows:

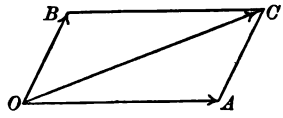
On OA, the *axis of real numbers*, we lay off $OB = +6$. From the end of the line, B, we erect a perpendicular and lay off upon it BC equal to 2 units, whence $BC = 2i$.

Then the sum of $+6$ and $2i$ is OC; that is, as with real numbers represented graphically, *the sum is the distance from the initial point to the terminal point.*



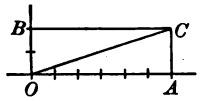
This does not mean that the absolute length of OC is equal to the sum of the absolute lengths of OB and BC, any more than we mean that the absolute length of $+1$ is equal to the sum of the absolute lengths of $+3$ and -2. In each case the question of direction is involved, and in each case we have to extend the original definitions of *sum* and *addition* to meet the conditions arising from the introduction of negative and imaginary numbers.

252. Addition of Forces. It is proved in physics that if we have a force pulling from O in the direction OA, and with an intensity expressed by the length OA; and another force also pulling from O, but in the direction OB, and with an intensity expressed by the length OB; then the result will be as if a force were pulling in the direction OC, and with an intensity expressed by the length OC.



This is commonly known in physics as the Law of the Parallelogram of Forces.

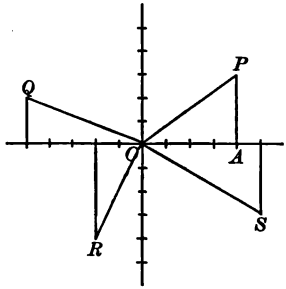
Similarly, if we have a force $OA = 6$, and a force $OB = 2i$, pulling at right angles to each other from O, the resulting force (or *resultant*) will be OC. We may therefore properly speak of OC as the sum of the forces OA and OB.



We therefore reach the same conclusion that was reached in § 251, that $6 + 2i$ may properly be represented graphically by the line OC.

Either by the method of the parallelogram of forces or by following the explanation given in § 251 we may represent graphically any complex number.

In the annexed figure $OP = 4 + 3i$, $OQ = -5 + 2i$, $OR = -2 - 4i$, and $OS = 5 - 3i$.



For purposes of plotting we may use squared paper, as in representing the graphs of equations. It must be understood, however, that when we are speaking of complex numbers the horizontal axis is the *real axis*, that is, the one on which real numbers are represented; while the perpendicular axis is the *imaginary axis*, that is, the one on which imaginaries are represented. This is not the case in the graphs of equations, for there we represent real numbers on both axes.

Exercise 113. Graphs of Complex Numbers

Represent the following by graphs:

1. $+7$.	11. $-2+3$.	21. $2+\frac{1}{2}i$.
2. -4.	12. $2+(-3)$.	22. $3\frac{1}{2}+i$.
3. $+3i$.	13. $2+3i$.	23. $-3\frac{1}{2}-i$.
4. $-4i$.	14. $2-3i$.	24. $-3+\frac{1}{2}i$.
5. $2.1i$.	15. $-2+3i$.	25. $2.5+2.5i$.
6. $3.5i$.	16. $-2-3i$.	26. $-3-3i$.
7. $-0.5i$.	17. $3+i$.	27. $6+2.5i$.
8. $-1.5i$.	18. $-3-i$.	28. $4+\sqrt{2}\sqrt{-1}$.
9. $\sqrt{2}\sqrt{-1}$.	19. $-3+i$.	29. $4+\sqrt{-3}$.
10. $\sqrt{-3}$.	20. $1-3i$.	30. $-2+\sqrt{-4}$.

31. Two forces are pulling in opposite directions, one to the right, with an intensity of 400 lb., and the other to the left, with an intensity of 300 lb. Represent graphically the resultant in direction and intensity.

32. Two forces are pulling at right angles to each other, one to the right, with an intensity of 3 lb., and the other upwards, with an intensity of 4 lb. Represent graphically the resultant in direction and intensity.

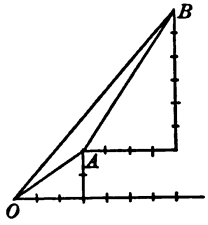
33. Two equal forces are pulling at right angles to each other, each with the intensity of $\sqrt{2}$. Represent graphically the resultant and tell the number of degrees in the angle that it makes with each of the two forces.

34. A ship is sailing at the rate of 15 mi. per hour. A man walks across the deck at right angles to the axis of the ship at the rate of 3 mi. per hour. Represent graphically the result of these two motions as a complex number.

35. Represent graphically the numbers 1, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$. What kind of a triangle seems to be formed by joining the three extremities that are remote from O?

253. Addition and Subtraction of Complex Numbers. If we wish to add $3 + 2i$ and $4 + 6i$, we may proceed as in ordinary addition and obtain $7 + 8i$ as the sum.

We may also proceed graphically, first drawing OA equal to $3 + 2i$; and then, from the point A, drawing AB equal to $4 + 6i$. The sum is, as usual, *the distance from the initial point (O) to the terminal point (B)*. That is, the line OB represents graphically the algebraic sum of the numbers represented by the lines OA and AB.



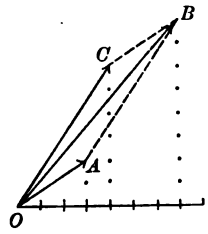
As in § 251, this does not mean that the absolute length of OB equals the sum of the absolute lengths of OA and OB, because there is also the question of direction to be considered. The sum of the absolute lengths of OA and AB is greater than the absolute length of OB, just as the sum of the absolute lengths of $+ 3$ and $- 2$ is greater than the absolute length of their algebraic sum.

The case of subtraction is included in that of addition, since we may merely change the signs of the subtrahend and add.

The graphic representation of the other operations requires a knowledge of trigonometry.

254. Addition of Forces. We may get the same result as in § 253 by considering the sum as the resultant of two forces.

Suppose the two forces $3 + 2i = OA$ and $4 + 6i = OC$ to act upon the point O as shown in the figure. Then the resultant will be OB, the diagonal of the parallelogram $OABC$.



We therefore have three plans for finding the sum of $3 + 2i$ and $4 + 6i$:

1. We may add as usual with binomials.

2. We may draw the first line; then, from its extremity, draw the second line; then connect the initial and terminal points.

3. We may represent the complex numbers as two forces and take their resultant.

All of these give the same result, of course, and all may be applied to real as well as complex numbers.

Exercise 114. Addition and Subtraction

Represent graphically the sums of the following:

1. $2+5i, 7+2i$.
2. $3+3i, 4+4i$.
3. $2-3i, 3+2i$.
4. $2-4i, 4-3i$.
5. $8-2i, 7-\frac{1}{2}i$.
6. $-3+i, 2+3i$.
7. $-4-i, 3+5i$.
8. $-5+i, 4-2i$.
9. $-5-2i, 4-3i$.
10. $-2-6i, 5+7i$.
11. $2-i, -3+i$.
12. $3-2i, -4-2i$.
13. $-7-3i, 6+2i$.
14. $-4-5i, 7+i$.
15. $-5-5i, 1-i$.

16. Represent graphically the sum of $2+3i$ and $4+i$. Then represent the sum of this result and $5+2i$.

17. As in Ex. 16 represent graphically the sum of $2+4i$, $3+2i$, and $3-2i$.

Represent graphically the sums of the following:

18. $2+i, 2-i, -4$.
19. $3+2i, 4-3i, -7+i$.
20. $2+5i, 3-2i, -2+6i$.
21. $2+0.5i, 3-0.5i, -5$.
22. $4+2i, 3-5i, -2+7i$.
23. $1, -\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

24. Four equal forces are pulling from a point, each at right angles to the two adjacent ones. Represent the resultant graphically.

25. A complex number is represented by a line that is inclined 45° to the real axis. Its absolute length is $5\sqrt{2}$. Write the complex number and represent graphically the result of adding it to $5-5i$.

26. By actual multiplication show that the six sixth roots of 1 are 1, $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, -1, $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, and $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$. Represent graphically each of these numbers and their sum.

27. Represent graphically the result of subtracting $2-3i$ from $4+2i$.

28. Represent graphically the result of subtracting $3+4i$ from the sum of $2+3i$ and $1+i$.

APPENDIX

255. Inequalities. In elementary algebra but little use is made of inequalities beyond a recognition of the meaning of $a > b$ and $a < b$. The following general principles may, however, be introduced:

1. *If unequals are operated on by positive equals in the same way, the results are unequal in the same order.*

That is, if $x = y$ (both being positive), and $a > b$, then

$$a + x > b + y \qquad ax > by \qquad a^x > b^y$$
$$a - x > b - y \qquad a \div x > b \div y \qquad \sqrt[x]{a} > \sqrt[x]{b}$$

For example, $2 > 1$ and $2 + 5 > 1 + 5$.

2. *If unequals are added to unequals in the same order, the sums are unequal in the same order.*

That is, if $x > y$ and $a > b$, then $x + a > y + b$.

3. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

That is, if $x = y$ and $a > b$, then $x - a < y - b$. This is seen in the case of $4 = 4$ and $3 > 2$, where $4 - 3 < 4 - 2$, since $1 < 2$.

The above three principles are used in geometry.

4. *The sum of the squares of two numbers is always greater than twice their product.*

For $(a-b)^2$, being a square, is always positive, and hence is greater than 0. Hence $a^2 - 2ab + b^2 > 0$, and therefore $a^2 + b^2 > 2ab$.

These four general principles should be illustrated by using numbers. It should also be shown that if unequals are subtracted from unequals, the result is indeterminate; that is, it may be an inequality in the same or reversed order, or it may be an equation. Thus, subtracting $4 > 1$, $3 > 1$, and $5 > 1$ from the inequality $5 > 2$, we have, respectively, $1 = 1$, $2 > 1$, $0 < 1$.

256. Cumulative Review. The following exercises are so arranged that at the close of each chapter all of the preceding work may be reviewed.

Exercise 115. Review of Chapter I

Using detached coefficients:

1. Add $23x^4 + 17x^3 - 14x^2 + 15x - 19$, $17x^2 - 20 + 15x^3 - 20x^4$.
2. Subtract $15x - 17 + 2x^3 - 7x^2$ from $13 - x^3 + 2x^2 - 7x$.
3. Multiply $12x^3 - 7x^2 - 9x + 2$ by $x^3 + 3x^2 - 2x + 1$.
4. Divide $x^4 + 5x^3 + 8x^2 + 5x + 1$ by $x + 1$.

Given $A = x^4 - 3x^3 - 21x^2 - 43x - 42$, $B = x + 2$, *and* $C = x - 7$, *perform the following operations:*

5. $A - (B + C)$.
6. $A + B - C$.
7. $A - BC$.
8. $BC - A$.
9. AB.
10. AC.
11. AB^2.
12. ABC.
13. $A \div B$.
14. $A \div C$.
15. $A \div (BC)$.
16. $A \div B \div C$.

Perform the following multiplications:

17. $x^3 - 1$ by $x^3 - 1$, $x^m - 1$ by $x^m - 1$.
18. $x^m + x^{m-1} + x^{m-2} + x^{m-3} + x^{m-4}$ by $x^m + 1$.
19. $(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$.
20. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^3 + 3a^2b + 3ab^2 + b^3$.
21. $a^5 + a^4 + a^3 + a^2 + a + 1$ by $a - 1$; by $a + 1$.

Perform the following divisions:

22. $a^4 - 3a^3b - 21a^2b^2 - 43ab^3 - 42b^4$ by $a - 7b$.
23. $x^4 - 2x^3 - 26x^2 - 57x - 42$ by $x^2 - 5x - 14$.
24. $x^{2m} + x^{m+1} + 2x^m - x - 3$ by x^{m-1}.
25. $x^m + 2x^{m-1} + x^{m-2} - 2x - 5$ by $x^{m-1} + x^{m-2} - 2$.
26. $a^{2x} + a^{2x-1} + a^{2x-2} + a^x + a^{x-1} + a^{x-2}$ by $1 + a^x$.

Exercise 116. Review of Chapters I–II

Perform the following operations, using detached coefficients when they make the work simpler, and checking each result:

1. $(x^7 + 3x^6 - 5x^5 + 2x^4 - 3x^3 + 6x^2 - 2x - 19)$
$+ (9x^7 - 3x^6 + 4x^5 - 2x^4 + 4x^3 - 5x^2 - 9x - 1)$
$+ (-10x^7 + x^5 - x^3 - x^2 + 11x + 20)$.
2. $(m^4 - 3m^3 + 7m^2 + 19m - 6) - (-m^4 - 3m^3 + 4m^2 - 6m)$.
3. $(P^2 - 17PQ - 9Q^2)(P^3 + 2P^2Q - 7PQ^2 + 2Q^3)$.
4. $(x^4 + 3x^2 - 29x + 7x^3 + 14) \div (3x - 2 + x^2)$.

Factor the following:

5. $4x^4 - 9$.
6. $16x^4 - 81$.
7. $x^6 - 64$.
8. $64x^6 - 729$.
9. $P^6Q^6 - 729R^6$.
10. $a^2b^2c^2 + 2abc - 63$.
11. $m^3 - 2m^2 - 143m$.
12. $m^4 + 4m^3 + 3m^2$.
13. $2P^2 + 9P - 45$.
14. $x^4 - 5x^3 + 6x^2$.
15. $6x^2 + 45x + 84$.
16. $14x^2 + 117x + 220$.
17. $14x^2 + 119x + 231$.
18. $P^4 + P^2Q^2 + Q^4$.
19. $x^4 + 4x^2 + 4$.
20. $(x^2 + 2)^2 - 4x^2$.
21. $a^4b^4 + 4$.
22. $49m^2 + 112m + 55$.
23. $56P^2 - 39PQ - 27Q^2$.
24. $a^2b^4c^6 - (a + b)^2$.
25. $x(63x + 130) + 63$.
26. $(a - b)^2 - (a + b)^2$.
27. $(a + b - c)^2 - (a - b + c)^2$.
28. $m^3 - 4m^2 - 19m - 14$.
29. $P^3 - 5P^2 - 13P - 7$.
30. $x^3 + x(7 - 5x) - 3$.
31. $-21 + x^3 - x(11x - 31)$.
32. $x^2(x - 18) + 95x - 126$.
33. $x(3 + x^2 + 2x) + x^2 + 1$.
34. $p^3 - 2p^2q - 57pq^2 - 54q^3$.
35. $a^3 + 5a^2 - 8a - 12$.
36. $x^3 + 9x^2 + 13x - 2$.
37. $P^3 - P^2 + 5P + 7$.
38. $P^3 + 12P^2 + 19P + 8$.
39. $M^3 + 6M^2 + 5M - 12$.
40. $M^3 - 13M + 12$.
41. $p^4 - p^3 - 3p^2 + 5p - 2$.
42. $p^4 + 6p^3 + 12p^2 + 10p + 3$.
43. $a^4 + 4a^3 + 6a^2 + 4a + 1$.
44. $a^5 + 5a^4 + 10a^3 + 10a^2 + 5a + 1$.

Exercise 117. Review of Chapters I–III

Perform the following operations:

1. $[x^7 + 2x - 3x^4 - 6x(x+2) + 5][x(x+7) - 3]$.
2. $a - 2b(c - d + \overline{a-b}) + ab - [2bd - (ab + b^2 - \overline{a+b})]$.
3. $x^3 - [x^2 + (x-1) + x^3 - (1-x)] + [x^2 - (\overline{x-1} \cdot \overline{x+1})]$.
4. $(x^7 + 7x^6 + 21x^5 + 15x^4 - 12x^3 + 3x^2 - 2x + 7) \div (x+7)$.

Factor the following:

5. $m^2 - 16m + 64$.
6. $m^3 - 126 + 95m - 18m^2$.
7. $p^3 + 11p^2 + 32p + 28$.
8. $x^3 - 11x^2 + 32x - 28$.
9. $m^{12} - n^{12}$.
10. $x^{16} - 1$.
11. $a^6b^6c^6 - c^6$.
12. $x^8 + x^4y^2 + y^4$.
13. $(a+b)^3 - (x+y)^3$ and $(a+b+c)^3 - (a-b+c)^3$.

Solve the following equations:

14. $(x+2)(x+3)(x-1) = x^3 + 5x + 6$.
15. $(x^5 - 1) \div (x-1) = x^3(x^3+1) + x^5(x^3+1) + x^2(x^2+1) + 3$.
16. $x^6 - 14x^3 + 41x + 56 = 0$.
17. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

Find the H.C.F. of the following:

18. $7p^2 + 21pq + 14q^2$, $14p^4 + 84p^3q + 154p^2q^2 + 84q^3$.
19. $x^3 + 21x^2 + 147x + 343$, $x^2 + 14x + 49$, $x^2 + 15x + 56$.

Find the L.C.M. of the following:

20. $x^3 - 1$, $x^2 - 1$, $2x^2 + 2x + 2$.
21. $2x^2 + xy - y^2$, $3x^2 + 4xy + y^2$, $14x - 7y$, $9x + 9y$.

Perform the operations indicated:

22. $\dfrac{16p^2 - 17p + 12}{12p^2 - 25p + 12} + \dfrac{27p^2 + 18p - 24}{12p^2 + 7p - 12} + \dfrac{25p^2 - 25p + 6}{20p^2 - 23p + 6}$.

23. $\dfrac{m^2 - 7m + 12}{m^2 + 5m + 6} \cdot \dfrac{m^2 + m - 2}{m^2 - 5m + 4} \div \dfrac{3m^2 - 7m - 6}{2m^2 + 5m - 3}$.

Exercise 118. Review of Chapters I–IV

Find the product of:

1. $x^{2n+1} - x^{n+1} - x^n + x^{n-1}$ and $x^{n+2} - x^2 - x + 1$.
2. $m^p + 3m^{p-2} - 2m^{p-1}$ and $2m^{p+1} + m^{p+2} - 3m^p$.

Find the quotient of:

3. $2a^{m+1} - 2a^{n+1} - a^{m+n} + a^{2n}$ by $a^n - 2a$.
4. $(a + 2b)^3 + (b - 3c)^3$ by $a + 3(b - c)$.

Factor the following:

5. $2x^2 + 5xy + 2y^2 - y(2y + x)$.
6. $x^4 - 2abx^2 - a^4 - a^2b^2 - b^4$.
7. $x^2 - 33x + 209 - (3x^2 + 37x - 19)$.
8. If $f(x) = x^2 - 16x + 63$, what values of x will make $f(x) = 0$?

Find the H.C.F. and L.C.M. of:

9. $12(x^4 - y^4)$, $10(x^6 - y^6)$.
10. $x^4 + xy^3$, $x^3y + y^4$, $x^4 + x^2y^2 + y^4$.
11. $4x^2 - 8x - 5$, $12x^2 - 4x - 65$.

Perform the following operations:

12. $\dfrac{2}{3a} - \dfrac{1}{2b} - \dfrac{2a+3}{6a^2} + \dfrac{1}{2x^2} + \dfrac{3a-2b}{6ab}$.
13. $\dfrac{2a^3x^7}{3b^3} \div \dfrac{5a^4b^5}{4c^4x^6} \cdot \dfrac{15b^2c^2}{4a^9x} \div \dfrac{25a^4x}{18ab^2c^3}$.
14. $\left(\dfrac{x^4 - y^4}{x^2 - y^2} \div \dfrac{x+y}{x^2 - xy}\right) \div \left(\dfrac{x^2 + y^2}{x - y} \div \dfrac{x+y}{xy - y^2}\right)$.
15. $\left(\dfrac{a^2 + b^2}{b} - a\right)\left(\dfrac{a^2 - b^2}{a^3 + b^3}\right) \div \left(\dfrac{1}{b} - \dfrac{1}{a}\right)$.
16. $\dfrac{6a^2 - a - 2}{8a^2 - 2a - 3} \cdot \dfrac{8a^2 - 10a + 3}{12a^2 + a - 6} \cdot \dfrac{12a^2 + 17a + 6}{6a^2 + a - 2}$.
17. $(a + b + c)^2 + (a + b - c)^2 + (b + c - a)^2 + (c + a - b)^2$.

Exercise 119. Review of Chapters I–V

Find the quotient of:

1. $a^2 - 2b^2 - 3c^2 + ab + 2ac + 7bc$ by $a - b + 3c$.
2. $81x^4 + 36x^2y^2 + 16y^4$ by $9x^2 - 6xy + 4y^2$.

Factor the following:

3. $a^4 + 64b^4$.
4. $a^5 - 32b^5$.
5. $m^6 - 64n^6$.
6. $a^2x^2 + 14abx + 33b^2$.
7. $x^3 + 9x^2 + 16x + 4$.
8. $x^3 + 5x^2 + 7x + 2$.

Find the H.C.F. and L.C.M. of:

9. $8a^2x^2 - 24a^2x + 16a^2$, $12ax^2y - 12axy - 24ay$.
10. $x^2 - 5x + 6$, $x^2 - 4x + 4$, $x^2 - 4$.

Simplify the following:

11. $$\frac{\dfrac{2x+y}{y} - \dfrac{y}{2x+y}}{\dfrac{x}{x+y} - \dfrac{x+y}{x}}.$$

12. $$\frac{\dfrac{1+x}{x^2+1} - \dfrac{1+x^2}{x^3+1}}{\dfrac{1+x^2}{x^3+1} - \dfrac{1+x^3}{x^4+1}}.$$

13. Study the rate of change of the total area of the six faces of a cube as the edge changes, by drawing the graph of the function $a = 6e^2$.

14. If $f(x) = x^2 + 4x + 4$, and $f'(x) = x + 2$, find the value of $f(x) + f'(x)$, $f(x) - f'(x)$, and $f(x) \div f'(x)$.

15. How much water must be added to a quart of a 30% solution of ammonia to reduce it to a 20% solution?

16. A train leaves A at 9 A.M. and travels 40 mi. an hour. Fifteen minutes later a second train follows the first on a parallel track at the rate of 51 mi. an hour. Where will the second train pass the first?

17. Solve the equation $\dfrac{6x+7}{3} - \dfrac{3}{x+2} = 2x + \frac{1}{2}$.

18. If $f(x) = x^3 + 3x^2 + 3x + 1$, find $f(1)$ and $f(x+1)$.

Exercise 120. Review of Chapters I–VI

Solve the following:

1. $\frac{3x}{5} - \frac{x}{6} = \frac{26}{15}$.

2. $\frac{x-2}{3x-5} = \frac{6}{19}$.

3. $\frac{3x-5}{2x+10} = \frac{2}{3}$.

4. $\frac{3(5x-3)}{2(4x+3)} = \frac{6}{5}$.

5. $\frac{5x-6}{5} - \frac{3x}{4} = \frac{x-9}{10}$.

6. $\frac{12-3x}{4} - \frac{3x-11}{3} = 1$.

7. A boatman goes 5 mi. in 45 min. when he rows with the tide. When he returns it takes him 1 hr. 30 min., rowing against a tide that is half as strong. What is the rate of the stronger tide?

Solve the following:

8. $6x + 5y = 46$
   $10x + 3y = 66$

9. $2x + 7y = 52$
   $3x - 5y = 16$

10. $4x + 9y = 79$
    $7x - 17y = 40$

11. $2x - 7y = 8$
    $-9x + 4y = 19$

12. $x = 16 - 4y$
    $y = 34 - 4x$

13. $5x = 2y + 78$
    $3y = x + 104$

14. If $f(x) = \frac{1}{x}$, find $f\left(\frac{1}{2}\right)$, $f(0.1)$, and $f\left(\frac{1}{x}\right)$.

15. Three times the greater of two numbers exceeds twice the less by 27, and the sum of twice the greater and five times the less is 94. Find the numbers.

16. Two passengers have together 400 lb. of baggage. One pays \$1.20 and the other \$1.80 for excess above the weight allowed. If all the baggage had belonged to one person, he would have paid \$4.50. How much baggage is allowed free?

17. A number is formed by two digits. The sum of the digits is six times their difference. The number itself exceeds six times the sum of its digits by 3. Find the number.

Solve the following:

18. $\dfrac{2x}{3}+\dfrac{y}{2}=10$

$\dfrac{y}{4}=\dfrac{5x-7}{19}$

19. $\dfrac{4+y}{3}=\dfrac{x}{4}$

$\dfrac{x-8}{4}=\dfrac{y}{5}$

20. $\dfrac{x+y}{3}+x=15$

$\dfrac{x-y}{5}+y=6$

21. $\dfrac{x-1}{8}+\dfrac{y-2}{5}=2$

$\dfrac{2x}{7}+\dfrac{2y-5}{21}=3$

22. In the triangle XYZ, the sum of angles X and Y is 70°, and the sum of angles Y and Z is 160°. How many degrees are there in each of the three angles?

23. A can do a piece of work in a days, B in b days, and C in $a+b$ days. How many days will it take them all working together to do the work? Discuss the result when $b=2a$; when $b=\frac{1}{2}a$; when $b=a$; when $a=0$.

24. A father is now f years old and his son is s years old. In how many years will the father be n times as old as the son? Discuss the result for $n>1$, $n<1$, $n=1$.

Solve by determinants:

25. $x+3y=32$

$x-7y=-38$

26. $x+y+z=16$

$3x-2y+5z=43$

$5x+3y+2z=40$

27. $5x+11y=42$

$x-y=2$

28. $3x+2y+z=10$

$2x+5y+3z=21$

$x+7y+z=18$

29. Solve the equation $3\{5[5(x-3)-3]-7\}=x(x-1)-3$.

Simplify the following:

30. $\dfrac{a^2+7a+12}{a^2+9a+20}$.

31. $\dfrac{7x^4-28x^3-35x^2}{7x^4-7x^3-140x^2}$.

32. $\dfrac{9(a+4b)^2}{11ab}-\dfrac{(5a-13b)^2}{11ab}+\dfrac{(4a+5b)^2}{11ab}$.

Exercise 121. Review of Chapters I–VII

Expand the following determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix}$ 2. $\begin{vmatrix} 2 & 0 & 3 \\ 5 & 0 & 7 \\ 6 & 0 & 9 \end{vmatrix}$ 3. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}$.

4. Given $f(x) = x^2 + 2x + 1$, draw a graph showing the rate of change of $f(x)$ as x changes.

5. If $f(x) = \frac{a}{x - a}$, discuss the nature of $f(x)$ when $x > a$, $x < a$, $x = a$, $a = 0$, $x = 0$, $x = 2a$.

6. Write illustrations, not already given in the book, of an imaginary number, a real number, a rational number, an irrational number, a complex number.

7. Expand $(x - 7y)^2$, $(\frac{1}{2}x - \frac{1}{3}y)^2$, $(\frac{1}{3}x - \frac{3}{4}y)^3$, $(x - \frac{1}{2}y)^5$.

8. Expand $(\frac{1}{2} - x)^5$, $(x - \frac{2}{3})^6$, $(2 + 3a)^6$, $(1 - \frac{1}{2}x)^8$.

Find the square root of:

9. 7225. 11. 4.8841. 13. 10.647169.

10. 546,121. 12. 3684.49. 14. 5347.265625.

Find the cube root of:

15. 85,184. 16. 681,472. 17. 2,985,984.

Find the square root of:

18. $4x^6 - 20x^4 + 28x^3 + 25x^2 - 70x + 49$.

19. $9x^6 + 12x^5y^2 - 2x^4y^4 - 34x^3y^6 - 19x^2y^8 + 10xy^{10} + 25y^{12}$.

Find the cube root of:

20. $x^6 - 15x^5y + 96x^4y^2 - 335x^3y^3 + 672x^2y^4 - 735xy^5 + 343y^6$.

21. $x^{12} + 9x^{10} + 12x^8 - 63x^6 - 60x^4 + 225x^2 - 125$.

22. Find the value of $r = \sqrt{\frac{2v}{\pi h} - \frac{h^2}{3}}$ for $v = 12.679$, $h = 1.2$, and $\pi = 3.14$.

Find the square root of:

23. $7+\sqrt{24}$. 25. $15-4\sqrt{11}$. 27. $60+16\sqrt{11}$.

24. $13-4\sqrt{3}$. 26. $16-4\sqrt{7}$. 28. $27-18\sqrt{2}$.

Perform the operations indicated:

29. $(7+2\sqrt{3})(7-2\sqrt{3})$. 31. $(5+2\sqrt{7})(3-5\sqrt{7})$.

30. $(2-\sqrt{-4})(2+\sqrt{-4})$. 32. $(8+\sqrt{-1})(9-3\sqrt{-1})$.

Rationalize the denominators of:

33. $\frac{1}{\sqrt{2}}$. 35. $\frac{a}{\sqrt{a}}$. 37. $\frac{1}{2\sqrt{5}}$. 39. $\frac{4}{\sqrt{28}}$.

34. $\frac{3}{\sqrt{5}}$. 36. $\frac{3a}{\sqrt{3}}$. 38. $\frac{5}{\sqrt[3]{5}}$. 40. $\frac{5}{\sqrt{5a}}$.

Simplify the following:

41. $(\sqrt[3]{m})^6$. 42. $\sqrt[n]{(x+y)^{2n}}$. 43. $3\sqrt{5}\cdot 5\sqrt{3}$.

Rationalize the denominators of:

44. $\frac{3}{\sqrt{5}+2}$. 45. $\frac{1}{\sqrt{5}+\sqrt{3}}$. 46. $\frac{1}{\sqrt{a}-\sqrt{b}}$.

47. Multiply both terms of the fraction $\frac{3}{\sqrt{4+\sqrt{7}}}$ by $\sqrt{4-\sqrt{7}}$ and simplify the result.

Rationalize the denominators of:

48. $\frac{5}{\sqrt{6-\sqrt{11}}}$. 49. $\frac{1}{\sqrt{5+2\sqrt{6}}}$. 50. $\frac{2}{\sqrt[3]{\sqrt{13}-\sqrt{5}}}$.

Simplify the following:

51. $(a^{n-3})^{-n}$. 53. $(100^{\frac{1}{3}})^{\frac{1}{2}}$. 55. $(x^{a^2-1})^{\frac{1}{a+1}}$.

52. $(27^{\frac{1}{2}})^{\frac{1}{3}}$. 54. $(144^{\frac{2}{3}})^{\frac{1}{2}}$. 56. $(x^{3n}y^{n^2}z^{-5n})^{\frac{1}{n}}$.

57. Raise $1+2\sqrt{-1}$ to the second power; the third power.

Exercise 122. Review of Chapters I–VIII

Perform the operations indicated and simplify the results:

1. $(3a^3 - 2a^2 + 3a - 5)(3a^3 + 2a^2 - 3a - 5)$.
2. $(x^4 - 2x^3 + x^2 - x)(x^4 + 2x^3 - x^2 - 2x)$.
3. $(5y^2 - y)^2 - (4y^2 + 5y + 1)^2 - (3y^2 + 5y + 1)^2$.
4. $(5a + 17)(5a - 17) - (4a - 15)(4a + 15)$.
5. $(2a - 1)^3 + (2a^2 - 3a + 1)^2 - (4a^2 + 1)(a - 1)a$.
6. $(2x^4 + 3x^3 - 14x^2 - 15x + 20) \div (x^2 - 5)$.
7. $(2a^3 - 9a^2x + 13ax^2 - 6x^3) \div (a^2 - 3ax + 2x^2)$.
8. $(6x^2 - 5xy + x - 6y^2 + 31y - 35) \div (3x + 2y - 7)$.

Factor the following:

9. $a^2 - b^2 + x(x + 2a)$.
10. $y^2 - 4z^2 + 5(2y + 5)$.
11. $2P^2 + 17P + 35$.
12. $15Q^2 - 2Q - 8$.

Simplify the following:

13. $(x^{-3} - 7x^{-1} - 6) \div (x^{-2} + 3x^{-1} + 2)$.
14. $\dfrac{3x}{7} - \dfrac{5x}{7} + \dfrac{x-1}{5} + \dfrac{2x}{7} - \dfrac{3x-2}{5} - \dfrac{11x}{7} + \dfrac{7x+4}{5} - \dfrac{3x}{7}$.
15. $(y^{-2} - 4x^{-1}y^{-1} - x^{-1}y^{-2} + 4x^{-2} + 2x^{-2}y^{-1} + x^{-2}y^{-2})(x - 2y + 1)$.
16. $(2ax^{-1} + a^{-1}x + 2)(5ax^{-1} + 3a^{-1}x - 1) - (2ax^{-1} + a^{-1}x + 2)(3ax^{-1} + 2a^{-1}x + 1)$.
17. $\dfrac{1}{1 - \dfrac{a}{a - \dfrac{1}{a + a^{-1}}}}$.
18. $\dfrac{\dfrac{1}{a-1} + \dfrac{1}{a+1}}{\dfrac{1}{a-1} - \dfrac{1}{a+1}}$.

Solve by determinants:

19. $x + 2y - z = 12$
    $2x - y + z = 0$
    $3x + 3y - 2z = 24$
20. $x + y + z = 0$
    $2x + y + 3z = -2$
    $3x - 2y - 2z = 10$

Exercise 123. Review of Chapters I–IX

Perform the following operations:

1. $(x^4 + 2x - 4x^{-2} + 8x^{-5} + 16x^{-8}) \cdot (x - 2x^{-2})$.
2. $(x^{-4} + 7x^{-3} + 12x^{-2} + 13x^{-1} - 3) \div (x^{-2} + 2x^{-1} + 3)$.
3. $(a^{-3} + 2a^{-2}x - a^{-1}x^2 - 2x^3) \cdot (a^{-3} - 2a^{-2}x + a^{-1}x^2 - 2x^3)$.
4. $(n^{-2} + n^{-4} + 25n^{-6}) \div (n^2 - 3n + 5)$.

Simplify the following:

5. $(a^{x-6r}y^{r+1})^{\frac{1}{r}} \cdot (a^{3r-4x}y^{r-3})^{\frac{1}{r}} \cdot (a^{4r+3x}y^{2-2r})^{\frac{1}{r}}$.
6. $(a^{5n-3}b^{7-8n})^{\frac{1}{n}} \cdot (a^{5-7n}b^{8n+1})^{\frac{1}{n}} \cdot (a^{3n-2}b^{6n-8})^{\frac{1}{n}}$.
7. $(\sqrt{x} + \sqrt{y} + 3)^2 - (\sqrt{x} + \sqrt{y} - 3)^2$.
8. $(1 - \sqrt{10} + \sqrt{5} - \sqrt{2})(1 - \sqrt{10} - \sqrt{5} + \sqrt{2})$.
9. $\sqrt[3]{2 - \sqrt{2 + \sqrt{3}}} \cdot \sqrt[3]{2 + \sqrt{2 + \sqrt{3}}} \cdot \sqrt[3]{2 + \sqrt{3}}$.

Solve the following:

10. $\dfrac{2(x-8)}{x-4} - \dfrac{x-4}{x+2} = \dfrac{x-8}{x-6}$.
11. $\dfrac{3}{x-1} - \dfrac{1}{x+1} = \dfrac{1}{x+2} + \dfrac{1}{x-6}$.
12. $x + \frac{1}{3}y = 5$
    $\frac{1}{2}x + y = 5$
13. $\frac{5}{7}x - \frac{3}{10}y = 9$
    $\frac{3}{7}x + \frac{9}{10}y = 27$

Solve by determinants:

14. $2x + 3y + 4z = 29$
    $3x + 4y + 5z = 38$
    $4x + 5y + 7z = 51$
15. $11x + 7y - 3z = 52$
    $7x + 9y - 5z = 34$
    $10x - 3y + 2z = 29$

Solve the equations:

16. $xy = 45$
    $\dfrac{x}{y} = 5$
17. $xy = 2a^2$
    $\dfrac{2y}{x} = a$
18. $(x+4)(y-1) = 27$
    $\dfrac{x+4}{y-1} = 3$

Exercise 124. Review of Chapters I–X

Given the following workshop formulas, solve as indicated:

1. $F = \frac{1}{2}(d + d')\pi N$, solve for N.

2. $s = 16.08\,t^2$, solve for t.

3. $V = \frac{1}{2}\pi r^2(h + h')$, solve for h; for h'; for r^2; for r.

Simplify the following:

4. $\sqrt{35 \cdot 21 \cdot 15}$. 5. $\sqrt[3]{50} \cdot \sqrt[3]{45} \cdot \sqrt[3]{12}$. 6. $\sqrt[3]{6\tfrac{3}{13}} \cdot \sqrt[3]{4\tfrac{1}{3}}$.

7. $\sqrt{\sqrt{x^2 + 2y^2} + \sqrt{x^2 - 2y^2}} \cdot \sqrt{\sqrt{x^2 + 2y^2} - \sqrt{x^2 - 2y^2}}$.

8. Divide $\frac{1}{8}a^4y^{-8} - \frac{3}{4}\frac{4}{5}a^3y^{-6} + \frac{9}{1}\frac{1}{2}a^2y^{-4} - \frac{8}{6}\frac{1}{0}ay^{-2} + \frac{1}{2}$ by $\frac{1}{3}a^2y^{-4} - \frac{2}{5}ay^{-2} + \frac{3}{4}$.

9. Multiply $\sqrt{-5} - \sqrt{-3}$ by $\sqrt{-5} + \sqrt{-3}$.

10. Express with positive indices: $\sqrt[-1]{\frac{1}{2}}$, $\sqrt[-1]{3}$, $\sqrt[-\frac{3}{2}]{512}$.

11. In the triangle ABC, $AB = 15$ in., $BC = \frac{1}{3}$ of the perimeter, and $CA = \frac{1}{4}$ of the perimeter. Find the length of BC and CA.

12. Of three pieces of iron, A, B, and C, it is known that B is 5 lb. lighter than C and 2 lb. heavier than A, while C weighs twice as much as A and B together. Find the weight of each of the three pieces.

13. From a point P on the circumference of a circle, three points A, B, and C start and move about the circumference, each at a uniform rate, A and B in the same direction and C in the opposite direction. A makes the circuit in 12 min., B in 20 min., and C in 15 min. When will A and C again be together? B and C? When will A be equidistant from B and C?

14. Given $s = ct + \frac{1}{2}pt^2$, find the value of t.

15. Prove that, if $a : b = p : q$, $a^2 + b^2 : \dfrac{a^3}{a+b} = p^2 + q^2 : \dfrac{p^3}{p+q}$.

16. If $(a+b+c+d)(a-b-c+d) = (a-b+c-d)(a+b-c-d)$, prove that a, b, c, and d are in proportion.

Exercise 125. Review of Chapters I–XI

1. Find the L.C.M. of $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$, and $x^3-8x^2+19x-12$.

Simplify the following fractions:

2. $\dfrac{x^4+2x^2+9}{x^4-4x^3+4x^2-9}$.

3. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

4. $\dfrac{a^3+2a^2+2a}{a^5+4a}$.

5. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

6. Multiply x^2-x+1 by $x^{-2}+x^{-1}+1$.

7. Divide $a^2-b^2-c^2-2bc$ by $\dfrac{a+b+c}{a+b-c}$.

8. Simplify $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)\div\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}\right)$.

Solve the following equations:

9. $\sqrt{x+y}+\sqrt{x-y}=\sqrt{a}$
$\sqrt{x^2+y^2}+\sqrt{x^2-y^2}=b$

10. $x^2+y^2-(x+y)=a$
$x^4+y^4+x+y-2(x^3+y^3)=b$

11. $x-y=a$
$x^5-y^5=b^5$

12. $\sqrt{x^2+y^2}+\sqrt{x^2-y^2}=2y$
$x^4-y^4=a^4$

13. Given $gt^2=\pi^2 l$, find the value of t in terms of g, π, and l.

14. Find the integer whose square added to its cube equals nine times the next higher integer.

15. Prove that, if four quantities are in proportion and the second is a mean proportional between the third and fourth, the third is a mean proportional between the first and second.

16. Two spheres of iron whose radii are r and R are melted and are then molded into a single sphere. Find the radius of the new sphere.

17. Given $l=ar^{n-1}$, find the value of r.

18. What is the 11th term in the expansion of $(a-b)^{15}$?

19. What is the middle term in the expansion of $\left(a+\dfrac{1}{a}\right)^8$?

Exercise 126. Review of Chapters I–XII

1. The sum of how many terms of the progression $1 + 3 + 5 + 7 + \cdots$ equals 1,234,321?

2. Find the sum of the first n terms of the arithmetical progression whose rth term is $2r - 1$.

3. Find the sum of the first ten terms of the series $2 - 2^2 + 2^3 - 2^4 + \cdots$.

4. Find the limit of the sum of the series $\frac{3}{2} - \frac{2}{3} + \frac{8}{27} - \cdots$, as the number of terms increases indefinitely.

5. Find the sum of the first n terms of the series $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \cdots$.

6. In a geometric progression show that the product of any two terms equidistant from a given term is constant.

7. Prove that the sum of the coefficients of the odd terms in the expansion of $(1 + a)^n$ is equal to the sum of the coefficients of the even terms.

8. What is the middle term in the expansion of $(a + x)^{10}$?

9. What is the rth term from the beginning in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$?

10. By what should $a^{\frac{1}{2}} - b^{\frac{3}{4}}$ be multiplied to have the product rational?

11. Multiply $3 + 2\sqrt{-2}$ by $5 - 3\sqrt{-2}$.

12. If $a^b = b^a$, prove that $\left(\frac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$.

13. Solve the equation $2^{x+1} + 4^x = 80$.

14. Solve the equation $\frac{x}{x-1} = \frac{3}{2} + \frac{x-1}{x}$.

15. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.

16. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$.

17. Prove that $\frac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})} = \frac{1}{5}\sqrt{15}$.

Exercise 127. Review of Chapters I–XIII

1. Multiply $a^{-2} + 2ab^{-1} - 3a^4b^{-2} - 4a^7b^{-3}$ by $-5a^{-3}b^2$.

2. Divide $3a^{-1}b - 6a^{-2}b^2 + 3a^{-3}b^3$ by $3a^{-3}b$.

Simplify the following:

3. $\{[(5)^{-1}]^{-3}\}^{-1}$. 5. $\{[(-2)^{-1}]^{-2}\}^2$. 7. $(-2a^{-1}b^3c^{-3})^{-4}$.

4. $(a^2b^{-3})^{-2}$. 6. $(ax^{-1})^{-3}$. 8. $(-7x^{-2}yz^{-5})^{-2}$.

9. Square $\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$ and $3 + \sqrt{2} - \sqrt{5}$.

Simplify the following:

10. $\sqrt{\dfrac{x^2+10x+25}{x^2-10x+25}}$. 11. $\sqrt{\dfrac{a^2-2a+1}{25b^2}}$.

12. Solve the equation $\dfrac{3}{x+1} - \dfrac{1}{x-1} = \dfrac{2}{x+5}$.

13. Given $a^x = \sqrt[r]{a}$, find the value of x.

14. Solve the equation $\sqrt{3x-8} + \sqrt{x+5} = \sqrt{7x+4}$.

15. The difference between the tenth and sixth terms of an arithmetic progression is 32, and the sum of the fifth and seventh terms is 100. Find the first term.

16. Find the limit of the sum of the infinite series $a, \dfrac{a}{3}, \dfrac{a}{9}, \dfrac{a}{27}, \cdots$.

Write the equation that results from taking the logarithms of both members of the following equations:

17. $x = ab$. 19. $p = a^n$. 21. $x = abcd$.

18. $b = m \div n$ 20. $r = \sqrt[n]{a}$. 22. $y = ac \div b$.

Compute the following by logarithms:

23. $3.42 \times 48.7 \times 0.72$.

24. $\dfrac{2.43 \times 86.4 \times 0.07}{144.3 \times 29.86}$.

25. $(9.7 \times 0.873 \times 42.08)^{\frac{2}{3}}$.

26. $\dfrac{3.142 \times 8.276 \times 0.009}{438.7 \times 9.007}$.

Exercise 128. Review of Chapters I–XIV

Perform the operations indicated:

1. $\frac{4a+7}{a+1}+\frac{7a-1}{a-1}-\frac{11a-8}{a+1}-\frac{2(a+2)}{a-1}+\frac{7a-15}{a+1}$.

2. $\frac{x(x+8a)}{x^3-a^3}+\frac{2x-3a}{x^2+ax+a^2}-\frac{3}{x-a}$.

3. $(x^3y^{-3}-2xy^{-2}+4y^{-1}-8x^{-1})\cdot(x+2y)$.

Solve the following:

4. $x:y=5:3$, $x-y=14$

5. $x:y=7:5$, $3y-x=104$

6. $x:y=5:3$, $5x-7y=44$

7. What amount of money will amount to K dollars in m months if invested at simple interest at $P\%$ a month?

8. If $i=\sqrt{-1}$, what is the value of i^7? i^{18}? i^{26}? i^{32}?

9. Simplify $(7+4i)(3+4i)-(7+5i)^2+(5+3i)^2$.

Represent graphically the following:

10. $7i$.

11. $-4.5i$.

12. $3-0.5i$.

13. $-7+3\frac{1}{2}i$.

14. $(2-7i)+(-5+2i)$.

15. $(5+2i)+(6-7i)$.

16. $(2+4i)+(3-9i)-(4+3i)-(-2-7i)+4+6i$.

Plot the following equations and solve:

17. $x+y=28$, $x-y=6$

18. $x+y=7$, $xy=12$

19. $x^2+y^2=17$, $xy=4$

20. $x^2+y^2=74$, $x^2-y^2=24$

21. $5x^2-9y^2=11$, $3x^2+2y^2=14$

22. $x+y^2=3$, $x^2+y=5$

23. Given the equation $\log x=2.1072$, solve for x.

24. Given the equation $x^{\log x}=1{,}000{,}000{,}000$, solve for x.

257. Table of Powers and Roots. The following table will be found helpful in plotting and solving equations:

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

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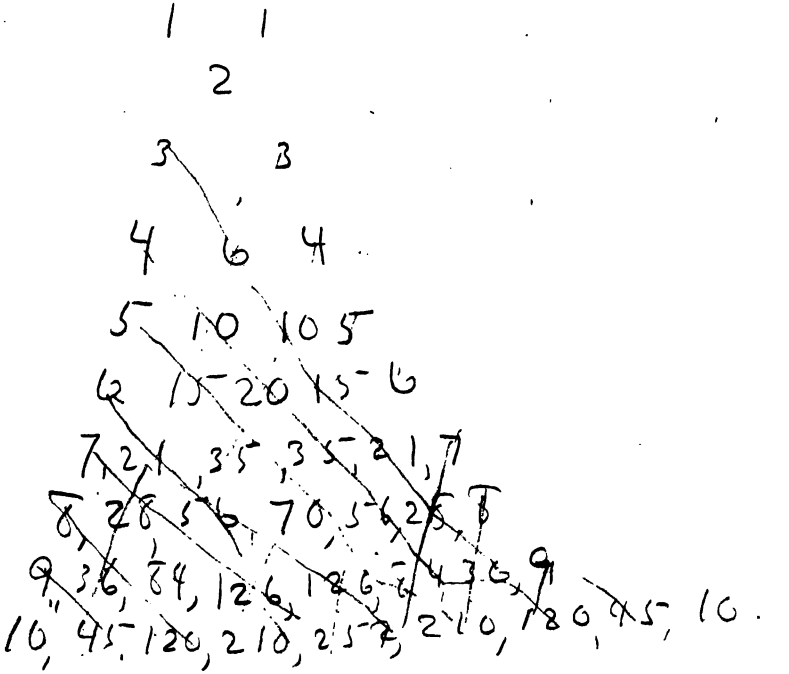
1 1
2
3 3
4 6 4
5 10 10 5
6 15 20 15 6
7 21 35 35 21 7
8 28 56 70 56 28 8
9 36 84 126 126 84 36 9
10 45 120 210 252 210 120 45 10

When I was a boy my
father said to me
I love you very
I wish o waer a Boy

$$a^{\frac{1}{2}} = \sqrt{a}$$

$$\sqrt{a} \div \sqrt{a^3}$$

John Adams

$$\begin{array}{r} 49 \\ 882 \\ \hline 931 \\ 22 \\ \hline 909 \end{array}$$

John Adams

John Adams

$$.09\frac{5}{16}$$

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